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## XXX. *A Theory of Elementary Particles.*—Part I.

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### 1. *Introduction.*

THE elementary particles, *i. e.*, the electron, the proton, and such entities, are now usually represented as combinations of a material mass with an

* Communicated by Sir W. H. Bragg, O.M., K.B.E., M.A., F.R.S.

electric charge distributed in a diffuse manner round a centre, and associated in a mysterious way with the so-called "material waves," which, in their turn, determine the distribution and the movement of these particles.

The visual picture of these cloudy electrical charges and associated material waves appears to be so uncertain that the branch of Theoretical Physics which has grown up as a result of their study—Wave Mechanics—has developed a definite tendency to dissociate itself from any visual picture whatsoever, and to concentrate on finding an adequate mathematical representation of the phenomena which are considered as manifestations of individual particles (" microscopic " phenomena), irrelevantly to any macroscopic models which can be quantitatively described, in full scale, by dimensions of a directly measurable order.

This renouncing of the macroscopic models would be the right way to follow, for the theory of microscopic phenomena, if the *only purpose* of science were to work but a convenient representation of quantitative relationships between the various sides of our experience, so as to enable us to foresee and to some extent control it; for by means of a well chosen system of symbols we can represent simultaneously, and in a compact and accurate way, much larger complexes than the most powerful imagination can put in a simultaneously contemplatable visual picture.

But the purpose of science is much wider: it is not only to enable us to foresee and to control the experience, but also to *understand* it, which means to construct from it a wide picture of the universe around us which would satisfy our mind.

To attain this satisfaction all our mental faculties must take an active part in the construction of this picture, and among them our concrete vision must occupy a position which would be adequate to its general importance in our mental life.

But our vision develops under the constant influence of our macroscopic experience, and so a picture of the universe around us, built of macroscopic models, would be for us more vivid and tangible than an abstract symbolic representation built by a skilful mathematician, however accurate and elegant his system of symbols may be.

Therefore the success of the abstract representation of the microscopic world, which has been already achieved by Wave Mechanics, and the continuous progress which it is making, must not be considered as a reason for ceasing to try to find the macroscopic models which would be suitable as bricks for building our picture of the world of elementary particles.

The laws governing the macroscopic models chosen for this purpose must be accurately known and expressible in a simple way. At the present stage of our knowledge the most suitable models in this respect will be those which are based on macroscopic electromagnetic phenomena in free space—in other words, models representing systems of Maxwell's electromagnetic waves in free space, of directly measurable dimensions.

The purpose of this paper is to construct the models of elementary particles in the form of systems of electromagnetic waves in free space.

If we succeed in our task, the electric "charge" and the material mass will cease to be separate entities, but may remain only as a sometimes convenient representation of certain systems of e.m. waves.

This paper will be divided into two parts.

In Part I. we shall derive the mathematical expressions for a system of e.m. waves, which we shall call "Electromagnetic Whirl," or simply "Whirl" for brevity.

It will be shown, *on the basis of Classical Electrodynamics*, that Whirls, to be stable, must satisfy the postulates of the Theory of Relativity and of Quantum Mechanics, and must form in their movement the e.m. waves satisfying the well-known de Broglie relationship between the speed of a particle and the speed of phase propagation of its waves.

The Whirls possess electromagnetic inertia, and the relationship between their masses and their energies exactly follows the well-known Einstein formula. They also possess angular momenta.

The quantum relationships in any form of movement of the Whirls or in radiation of their energy follow from the laws of conservation of energy and angular momentum.

Among the Whirls we shall be able to distinguish two categories depending on the relationship of the electric and the magnetic vectors in their electromagnetic field,

or, in other words, on their "state of polarization." We shall call these Whirls "electropolarized" and "magnetopolarized." The exact meaning of these terms will be explained later.

If two Whirls are polarized in the same way, we shall call them "copolarized." If their state of polarization lie at opposite extremities, we shall call them "contrapolarized."

In Part II. we shall consider the e.m. field formed by a system of Whirls, their relative positions, and their mutual forces.

We shall consider in detail a system of two Whirls. We shall find that their mutual distance forces depend, on the one hand, on the ratio of their dimensions, and on the other, on the relationship between their states of polarization.

On the average, and when the distance between the centres of the two Whirls is not too small, copolarized Whirls of similar structure repel each other if their sizes are equal, and attract each other if the dimensions of the one are much larger than the dimensions of the other. In both cases the attraction and the repulsion follow, again on the average, the inverse square law.

The force of attraction between unequal copolarized Whirls depends on the ratio between their masses. At a certain numerical value of this ratio this force equals the force of repulsion between identical Whirls. This numerical value happens to coincide, within the close limits of the accuracy of calculation, with the observed mass ratio of the proton and the electron.

The contrapolarized Whirls of similar structure and of equal dimensions practically do not interact unless the distance between their centres is very small, but if their dimensions greatly differ they attract each other, also in accordance with the inverse square law, but with a force considerably smaller than the copolarized Whirls of the same relative dimensions.

On the basis of these findings a theory of elementary particles, such as electrons, protons, positrons, and light quanta (photons) is outlined.

Their electromagnetic fields are defined quantitatively.

The electrons and the protons are considered as electropolarized Whirls, the positrons and the light quanta of cosmic rays as magnetopolarized ones.

It is suggested that the great velocity of particles in the cosmic rays may be only apparent, and their penetrating power, as well as the great radius of their curvature in the magnetic field, may be due to the relatively small value of the mutual forces of the contrapolarized Whirls.

Gravitation can be explained by the interaction of contrapolarized Whirls.

It appears that by this theory the postulates of Wave Mechanics and of the Principle of Relativity can be explained on the basis of the Maxwell equations. Moreover, on the basis of this theory the possibility is anticipated of the existence of many forms of electromagnetic particles, which can be indirectly discovered in future (like neutrons), and which are also Whirls, and, *ipso facto*, are forms of Maxwell electromagnetic waves.

## Part I.—The Electromagnetic Whirl.

### 2. *Rotating Cylindrical Waves* *.

One of the solutions of the Maxwell equations for free space can be expressed in cylindrical coordinates $(x, r, \theta)$ in the following way:

$$\left.\begin{matrix} E_x = \underset{\theta}{\mathrm{E}_x} e^{i\phi}, \end{matrix}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.1)$$

where

$$\phi = \Omega t + \kappa x + n\theta, \quad . \quad . \quad . \quad . \quad . \quad (2.2)$$

and

$$\left.\begin{aligned} \mathrm{E}_x &= -\zeta(\mathrm{A}_1 + \mathrm{A}_2)\mathrm{J}_n(\kappa\zeta r), \\ \mathrm{E}_r &= i(\mathrm{A}_1\mathrm{J}_{n+1}(\kappa\zeta r) - \mathrm{A}_2\mathrm{J}_{n-1}(\kappa\zeta r)), \\ \mathrm{E}_\theta &= \mathrm{A}_1\mathrm{J}_{n+1}(\kappa\zeta r) + \mathrm{A}_2\mathrm{J}_{n-1}(\kappa\zeta r). \end{aligned}\right\} \quad . \quad (2.3)$$

Here $E_x$, $E_r$, and $E_\theta$ are the components of the force of electric field; $i = \sqrt{-1}$; $e$ is the Napierian number, $\phi$ the phase; $\Omega$, $\kappa$, and $n$ are the phase constants, out of which $\Omega$ and $\kappa$ can have any value, while $n$ must be a real integer; $\mathrm{E}_x$, $\mathrm{E}_r$, and $\mathrm{E}_\theta$ are the amplitude components depending only on $r$; $\mathrm{A}_1$ and $\mathrm{A}_2$ are the amplitude constants; $\mathrm{J}_n(\ )$ is, according to the usual

* This section is a brief summary of the results of the author's previous paper in the Phil. Mag. (May 1935) adapted to the requirements of subsequent discussion in the present paper.

notation, a Bessel coefficient of the order $n$ of the expression in ( ), and

$$\zeta=\sqrt{\frac{\Omega^2}{\kappa^2 c^2}-1}. \quad . \quad . \quad . \quad . \quad . \quad (2.4)$$

We shall denote by R, before an expression, that we take only the real part of this expression.

Then

$$c_2=-\frac{\mathrm{R}\dfrac{\partial\phi}{\partial t}}{\mathrm{R}\dfrac{\partial\phi}{\partial x}}=-\frac{\mathrm{R}\,\Omega}{\mathrm{R}\,\kappa} \quad . \quad . \quad . \quad . \quad . \quad (2.5)$$

is the speed of propagation of the phase along the axis $x$, which we shall call the "axial speed of propagation of the waves"; $c$ is the speed of light.

The waves expressed by the above formulæ are axially symmetrical. They can be considered as *propagating along* the axis of symmetry (axis $x$) with the velocity $c_2$ and at the same time *rotating about* this axis with the angular velocity

$$\Omega_x=-\frac{\mathrm{R}\dfrac{\partial\phi}{\partial t}}{\mathrm{R}\dfrac{\partial\phi}{\partial\theta}}=-\frac{\mathrm{R}\,\Omega}{n} \quad . \quad . \quad . \quad . \quad . \quad (2.6)$$

Therefore we call these waves "rotating cylindrical waves."

The sign of the integer $n$ depends on the direction of rotation. Its absolute value shows the number of periods of the e.m. field in the angular direction, and is called therefore "the number of pairs of poles" of the rotating cylindrical waves, by analogy with the polyphase electric motors, which they recall by the shape of their field.

In a particular case, when $n=0$, the waves are no longer rotating, but simply propagating, and then they are called simply "cylindrical" waves instead of "rotating" ones.

The magnetic vector, the components of which we shall denote by $B_x$, $B_r$, and $B_\theta$, with the respective amplitudes $\mathrm{B}_x$, $\mathrm{B}_r$, and $\mathrm{B}_\theta$, can be expressed by the same formulæ as the electric one if the constants $\mathrm{A}_1$ and $\mathrm{A}_2$ are replaced by $i\mathrm{A}_1'$ and $i\mathrm{A}_2'$, where

$$\left.\begin{aligned}\mathrm{A}_1'&=\frac{1}{\sqrt{\zeta^2+1}}\left[\mathrm{A}_1+(\mathrm{A}_1+\mathrm{A}_2)\frac{\zeta^2}{2}\right],\\ \mathrm{A}_2'&=-\frac{1}{\sqrt{\zeta^2+1}}\left[\mathrm{A}_2+(\mathrm{A}_1+\mathrm{A}_2)\frac{\zeta^2}{2}\right].\end{aligned}\right\} \quad . \quad . \quad (2.7)$$

It can be seen from (2.1) and (2.3) that the end of the electric vector is moving on the surface of an ellipsoid, with the half-axes $|E_x|$, $|E_r|$, and $|E_\theta|$. We say, therefore, that the rotating cylindrical waves are, generally speaking, "ellipsoidally polarized" *.

In a particular case when

$$A_1=-A_2=A_e \quad . \quad . \quad . \quad . \quad . \quad (2.8)$$

we shall have from (2.3)

$$\left.\begin{aligned} E_x&=0,\\ E_r&=iA_e(J_{n+1}(\kappa\zeta r)+J_{n-1}(\kappa\zeta r))=\frac{i2nA_e}{\kappa\zeta r}J_n(\kappa\zeta r),\\ E_\theta&=A_e(J_{n+1}(\kappa\zeta r)-J_{n-1}(\kappa\zeta r)), \end{aligned}\right\} \quad (2.9)$$

and consequently the electric vector of the rotating cylindrical waves will be *elliptically* polarized in the plane $r\theta$.

At a large distance from the axis of symmetry ($\kappa\zeta r>>1$) $E_r$ will be relatively very small and $E\cong E_\theta$.

We shall call such waves "electropolarized" rotating cylindrical waves.

When $A_1=A_2=A_m$ we shall have from (2.3)

$$\left.\begin{aligned} E_x&=-2\zeta A_m J_n(\kappa\zeta r),\\ E_r&=\ iA_m(J_{n+1}(\kappa\zeta r)-J_{n-1}(\kappa\zeta r)),\\ E_\theta&=\ A_m(J_{n+1}(\kappa\zeta r)+J_{n-1}(\kappa\zeta r))=\frac{2nA_m}{\kappa\zeta r}J_n(\kappa\zeta r). \end{aligned}\right\} \quad (2.10)$$

We shall call such waves "magnetopolarized" rotating cylindrical waves.

It will be easily found from (2.7) that when $A_1=-A_2=A_e$, $A_1'=A_2'=A_e'$; and when $A_1=A_2=A_m$,

* When the present paper was already in print, I noticed that this interpretation of formulæ (2.1) and (2.3) was not quite correct. *Even in a general case* the end of the vector describes an *ellipse*. When $A_1$ and $A_2$ are real, this ellipse is situated in a plane perpendicular to the $x\theta$ plane, and its half axes are $|E_r|$ and $\sqrt{|E_x|^2+|E_\theta|^2}$. The line of intersection between the plane of this ellipse and the $x\theta$ plane forms with the axis $x$ the angle $\tan^{-1}\frac{E_x}{E_\theta}$. This correction makes the term "ellipsoidal," as distinct from the term "elliptical" (when the ellipse of polarization lies in $r\theta$ plane), only conventional. Otherwise it is of no consequence for the present paper, as well as for the previous one, *loc. cit.* This footnote should be read in conjunction with p. 943 of Phil. Mag., May 1935, where the term "ellipsoidal polarization" was first mentioned.

$A_1' = -A_2' = A_m'$. Consequently formulæ (2.10) represent the electric vector of rotating cylindrical waves with an *elliptically* polarized *magnetic* vector.

From the same formulæ we can find the expressions for the amplitude constants of the magnetic vector in the elliptically polarized rotating cylindrical waves in terms of the corresponding amplitude constants of the electric vector

$$A_e' = \frac{A_e}{\sqrt{\zeta^2+1}}, \qquad A_m' = A_m\sqrt{\zeta^2+1}. \quad . \quad (2.11)$$

It is not difficult to see that any rotating cylindrical wave-train can be resolved into two component elliptically polarized wave-trains, one electropolarized and the other magnetopolarized.

The relationship between the amplitude constants of these component wave-trains and the amplitude constants of the resultant ellipsoidally polarized wave-train will be as follows :—

$$\left.\begin{aligned} A_e &= \frac{A_1 - A_2}{2} = \frac{A_1' + A_2'}{2}\sqrt{\zeta^2+1}, & A_m &= \frac{A_1 + A_2}{2} = \frac{A_1' - A_2'}{2\sqrt{\zeta^2+1}}, \\ A_1 &= A_m + A_e, & A_1' &= A_e' + A_m', \\ A_2 &= A_m - A_e, & A_2' &= A_e' - A_m'. \end{aligned}\right\} \quad (2.12)$$

When one of the vectors, the electric or the magnetic, is elliptically polarized we shall call it a " transversal " vector, and the other (the ellipsoidally polarized one) the " longitudinal " vector.

The shape of the lines of force of the transversal and of the longitudinal vectors is independent of whether the electric vector is the transversal one and the magnetic is the longitudinal one, or *vice versâ*.

### 3. *Rotating Cylindrical Waves with a Complex Phase.*

The phase constants $\Omega$ and $\kappa$, which are, generally speaking, complex quantities, can be written in the form

$$\Omega = \Omega_1 + i\Omega_2, \qquad \kappa = \kappa_1 + i\kappa_2, \quad . \quad . \quad . \quad (3.1)$$

where $\Omega_1$, $\Omega_2$, $\kappa_1$, and $\kappa_2$ are real constants. Then the phase $\phi$ can be expressed in the form

$$\phi=\phi_1+i\phi_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.2)$$

where

$$\left.\begin{aligned}\phi_1&=\Omega_1 t+\kappa_1 x+n\theta\\ \text{and}\qquad \phi_2&=\Omega_2 t+\kappa_2 x\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3.3)$$

are real. $\phi_1$ will be called the "real" component and $\phi_2$ the "imaginary" component of the phase $\phi$.

By (2.1) and (3.2)

$$\left.\begin{matrix}E_x\\ r\\ \theta\end{matrix}=\begin{matrix}\mathrm{E}_x e^{-\phi_2+i\phi_1}\\ r\\ \theta\end{matrix}\right\} . \quad . \quad . \quad . \quad . \quad . \quad (3.4)$$

This gives an expression for rotating cylindrical waves with a complex phase.

By (2.5), (3.2), and (3.3) the speed of propagation

$$c_2=-\frac{\Omega_1}{\kappa_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.5)$$

and by (2.6) the angular velocity of rotation

$$\Omega_x=-\frac{\Omega_1}{n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.6)$$

In order that the expressions (2.3) for the amplitude of rotating cylindrical waves with a complex phase should have a physical meaning the independent variable of the Bessel coefficients must be real, for otherwise, by the well-known property of Bessel Functions, the amplitude will tend to infinity when $r\to\infty$.

Hence $\kappa\zeta$ must be real, and in the expression, drawn from (2.4) and (3.1),

$$\kappa\zeta=\sqrt{\frac{\Omega^2}{c^2}-\kappa^2}$$

$$=\sqrt{\frac{{\Omega_1}^2-{\Omega_2}^2}{c^2}-{\kappa_1}^2+{\kappa_2}^2+2i\left(\frac{\Omega_1\Omega_2}{c^2}-\kappa_1\kappa_2\right)}, \quad (3.7)$$

we must have

$$\frac{\Omega_1\Omega_2}{c^2}-\kappa_1\kappa_2=0,$$

or
$$c^2=\frac{\Omega_1}{\kappa_1}\frac{\Omega_2}{\kappa_2}=-c_2\frac{\Omega_2}{\kappa_2}, \text{ by (3.5)}, \quad . \quad . \quad (3.8)$$

whence

$$\kappa\zeta=\sqrt{\frac{\Omega_1^2-\Omega_2^2}{c^2}-\kappa_1^2+\kappa_2^2}=\frac{1}{\beta}\sqrt{\kappa_2^2+\kappa_0^2}$$

$$=\sqrt{q^2+q_0^2}, \quad \ldots\ldots\ldots\ldots \quad (3.9)$$

and, by the second formula (3.1),

$$\zeta=\frac{\sqrt{\kappa_2^2+\kappa_0^2}}{\beta(\kappa_1+i\kappa_2)}=\frac{\sqrt{\kappa_2^2+\kappa_0^2}}{\beta\sqrt{\kappa_1^2+\kappa_2^2}}e^{-i\tan^{-1}\frac{\kappa_2}{\kappa_1}}$$

$$=\frac{\sqrt{q^2+q_0^2}}{\beta\sqrt{q^2+q_1^2}}e^{-i\tan^{-1}\frac{q}{q_1}}=\frac{1}{\beta}\frac{\sqrt{q^2+q_0^2}}{q_1+iq}, \quad (3.9a)$$

where

$$\beta=\frac{1}{\sqrt{1-\frac{c^2}{c_2^2}}}, \quad \ldots\ldots \quad (3.10)$$

and

$$\kappa_0=-\frac{\Omega_1}{c}=\frac{2\pi}{\lambda_0}, \quad \ldots\ldots \quad (3.10a)$$

where $\lambda_0$ is the wave-length of light waves at the given frequency $\Omega_1$,

$$q=\frac{\kappa_2}{\beta}, \quad q_0=\left|\frac{\kappa_0}{\beta}\right|, \quad q_1=\frac{\kappa_1}{\beta}. \quad \ldots \quad (3.11)$$

It can be seen from (3.3) and (3.4) that if $\kappa_2$ keeps a constant value, $E$ will become infinity either at $x=+\infty$ or at $x=-\infty$, depending on the sign of $\kappa_2$.

In order to avoid this difficulty we shall substitute for $x$ in both formulæ (3.3), the absolute value of $x$, which we shall denote in the usual way by $|x|$. Hence instead of (3.3) we shall have

$$\left.\begin{aligned}\phi_1&=\Omega_1 t+\kappa_1|x|+n\theta=\Omega_1\left(t-\frac{|x|}{c_2}\right)+n\theta,\\ \phi_2&=\Omega_2 t+\kappa_2|x|=\left(|x|-\frac{c^2}{c_2}t\right)\kappa_2=\beta\left(|x|-\frac{c^2}{c_2}t\right)q.\end{aligned}\right\} \quad (3.12)$$

It will be easily seen from (3.4) that this substitution of $|x|$ for $x$ does not introduce any discontinuity in the e.m. field.

Recalling the general definition of the speed of propagation of cylindrical waves given in (2.5) we can see that the first equation (3.12) shows that our waves propagate *in both directions* from the central plane.

It follows from the discussion in this section that such is the only physically possible form of the rotating cylindrical waves with a complex phase in the case where the field increases with time. If the field were decreasing then, as it is not difficult to see, the waves would propagate *towards* the central plane.

It will be noted that $c_2$ in (3.5) no longer expresses the speed of propagation of our waves generally, but only the speed on one side of the central plane. On the opposite side this speed is equal $-c_2$.

The factor $e^{-\phi_2+i\phi_1}$ in (3.4), with $\phi_2$ expressed according to the second equation (3.12), can be interpreted as indicating that the waves are only partially propagating into space, and partially* are reflected backwards.

4. *Stability Conditions of a Rotating Cylindrical Wave-Train with a Complex Phase. The Electromagnetic Whirl.*

The waves which were discussed in the previous section must, generally speaking, have an exponential factor decreasing (or increasing) with time. There is, however, one exception, namely, when $c_2=\infty$ , or $\kappa_1=0$, *i. e.*, $\lambda=\infty$ (where $\lambda$ is the wave-length). In this case we shall have by (3.12),

$$\left.\begin{aligned}\phi_1&=\Omega_1 t+n\theta,\\ \phi_2&=\beta q|x|=\kappa_2 x|,\ \text{by (3.11)},\end{aligned}\right\}\quad . \quad . \quad (4.1)$$

In such a wave-train there will be no *aperiodic time variation*, but, on the other hand, there will be *no periodic variations with respect to x*. The waves will only *rotate* about the axis of symmetry, but there will be no propagation of the waves along this axis. When $n=0$, *i. e.*, in the case of simply cylindrical waves, but not rotating ones, the waves with the expression for the phase in accordance with (4.1) will not rotate but pulsate.

The position will be different if the whole wave-train with its central plane moves in the axial direction, say, with a speed $v_x$.

Then, referring our equations to the co-axial cylindrical coordinate system which has its origin in the moving central plane (" moving " coordinate system), and *retaining our notation x* for this *moving* coordinate system,

* Compare Frankel, ' Wave Mechanics, Advanced General Theory,' Oxford, 1934.

we must substitute in our previous equations $x+v_x t$ and $|x|+v_x t$ for $x$ and $|x|$ resp.*. Hence, with reference to the moving coordinate system, (3.12) must be written

$$\left.\begin{aligned}\phi_1&=\Omega_1\left(t-\frac{|x|+v_x t}{c_2}\right)+n\theta,\\ \phi_2&=\beta\left(|x|+v_x t-\frac{c^2}{c_2}t\right)q.\end{aligned}\right\} \quad . \quad . \quad . \quad (4.2)$$

In order that this moving wave-train should not undergo any aperiodic time variation we must obviously have, by (3.4) and second formula (4.2),

$$c_2 v_x = c^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.3)$$

which is the well-known relationship which de Broglie assumed for his material waves.

By (4.3) and (3.10),

$$\beta=\frac{1}{\sqrt{1-\frac{v_x^2}{c^2}}}, \quad . \quad . \quad . \quad . \quad . \quad (4.4)$$

which is the well-known coefficient from the Lorenz-Einstein transformations.

By (4.2), (4.3), and (4.4),

$$\left.\begin{aligned}\phi_1&=\frac{\Omega_1}{\beta}\left(\frac{t}{\beta}-\frac{\beta|x|v_x}{c^2}\right)+n\theta,\\ \phi_2&=\beta|x|q.\end{aligned}\right\} \quad . \quad . \quad . \quad (4.5)$$

We can thus see that in order that a rotating cylindrical wave-train with a complex phase should remain stable (*i. e.*, should not undergo any aperiodic time variation of the amplitude) during the uniform translatory motion along the axis of symmetry, it is necessary that the axial distribution of the field should undergo a definite change depending on the speed of translatory motion. From purely exponential it must be converted into exponentially periodic, and instead of pure rotation (or pulsation, in the case of $n=0$), which we had for a stable wave-train at rest, we shall have not only rotation of the wave-train about the axis (or pulsation), but also a propagation along it.

* If we write $|x+v_x t|$ instead of $|x|+v_x t$, it would mean that we must always so choose the positive direction of $x$ as to have $v_x > 0$, otherwise it would not make any difference.

We shall find now the conditions of "stability of configuration," *i. e.*, the conditions under which the change in the stable rotating cylindrical wave-train associated with its uniform translatory movement, will be *restricted to the redistribution of the field along the axis, while the radial and azimuthal distribution of the electromagnetic field will remain unaltered*, in other words, when the change of the field structure will be reduced to a minimum.

We can see from (2.3) that the necessary condition for retaining the radial distribution of the field unaltered (apart from a common amplitude constant), is

$$\kappa\zeta = \text{const.}, \quad . \quad . \quad . \quad . \quad . \quad (4.6)$$

whence, by (3.9) and (3.10)–(3.11),

$$\left.\begin{aligned} q &= \frac{\kappa_2}{\beta} = \text{const.}, \\ q_0 &= \left|\frac{\Omega_1}{\beta c}\right| = -\frac{\Omega_0}{c} = \text{const.}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4.7)$$

where $\Omega_0$ is the angular frequency at rest ($\beta=1$), and hence

$$\Omega_1 = \beta\Omega_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.8)$$

which means that the frequency must be proportional to $\beta$.

By (4.8), (4.3), and (3.5),

$$\left.\begin{aligned} c_2 &= -\frac{\Omega_1}{\kappa_1} = -\frac{\beta\Omega_0}{\kappa_1} = \frac{c^2}{v_x}, \\ \text{or} \\ \kappa_1 &= -\frac{\beta\Omega_0 v_x}{c^2} = \pm\beta q_0 \frac{v_x}{c}, \text{ by (4.7)}, \end{aligned}\right\} \quad . \quad . \quad (4.9)$$

whence recalling that $\Omega = -2\pi\nu$, where $\nu$ is the frequency, while $\kappa_1 = \frac{2\pi}{\lambda}$, where $\lambda$ is the wave-length, and denoting by $\nu_0 = -\frac{\Omega_0}{2\pi}$ the frequency of the wave-train at rest, we have

$$\lambda = \frac{c^2}{\beta\nu_0 v_x}, \quad . \quad . \quad . \quad . \quad . \quad (4.10)$$

by (4.8).

This is nothing else but the well-known de Broglie formula for the wave-length of "material" waves *.

By (4.9), (3.11), (3.10), and (3.9 *a*), we have

$$\zeta=\frac{\sqrt{q^2+{q_0}^2}}{\beta\sqrt{q^2+{q_0}^2\frac{{v_x}^2}{c^2}}}e^{-i\tan^{-1}\frac{qc}{q_0v_x}}$$

$$=\frac{\sqrt{q^2+{q_0}^2}}{\sqrt{\beta^2q^2+{q_0}^2(\beta^2-1)}}e^{-i\tan^{-1}\frac{\beta q}{q_0\sqrt{\beta^2-1}}}=\frac{\sqrt{q^2+{q_0}^2}}{i\beta q+q_0\sqrt{\beta^2-1}}; \quad \ldots \quad (4.10\,a)$$

$$\sqrt{\zeta^2+1}=\frac{\beta q_0+iq\sqrt{\beta^2-1}}{q_0\sqrt{\beta^2-1}+iq\beta}$$

$$=\frac{\sqrt{\beta^2{q_0}^2+q^2(\beta^2-1)}}{\sqrt{\beta^2q^2+{q_0}^2(\beta^2-1)}}e^{-i\tan^{-1}\frac{\beta q}{q_0\sqrt{\beta^2-1}}+i\tan^{-1}\frac{q\sqrt{\beta^2-1}}{q_0\beta}} \quad \ldots \quad (4.10\,b)$$

It will be borne in mind that, by definition, $c_2$, $\kappa_1$, and $\Omega_1$ *relate to the observer at rest.*

By (4.7)–(4.9), the equations (4.5) can be written in the form

$$\left.\begin{aligned}\phi_1&=\Omega_0\left(\frac{t}{\beta}-\frac{\beta|x|v_x}{c^2}\right)+n\theta\\&=c\left(t'-\frac{v_x}{c^2}\xi\right)+n\theta,\\\phi_2&=\beta q|x|=\chi\xi,\end{aligned}\right\} \quad \ldots \quad (4.11)$$

where

$$\left.\begin{aligned}t'&=\frac{\Omega_0}{\beta c}t=q_0\frac{t}{\beta},\quad \xi=\beta q_0|x|,\\\chi&=\frac{q}{q_0}=\frac{\kappa_2}{\beta q_0}.\end{aligned}\right\} \quad \ldots \quad (4.12)$$

These equations determine the dependence on $x$ and $t$ of the e.m. field of the rotating electromagnetic wave-train with an exponentially variable amplitude factor, which does not suffer, during its uniform translatory

* This is more usually written in the form $\lambda=\frac{h}{m_0\beta v_x}$, where $h$ is the Planck constant, and $m_0$ is the "mass" at rest, but since $m_0c^2=h\nu_0$, this is equivalent to the formula (4.10).

motion in the axial direction, either any aperiodic change with the time, or any alteration of the relative radial or azimuthal distribution of the e.m. field.

We shall call such a wave-train an "Electromagnetic Whirl," or simply "Whirl," for brevity.

### 5. *The Influence of Uniform Axial Motion on the Amplitude Constants of the Whirl.*

In order to maintain the "stability of configuration"* of a Whirl in uniform translatory motion in the direction of its axis of symmetry, certain relationships between the speed of this motion and the amplitude constants must be observed, apart from the relationships between this speed and the phase distribution, which we have found in the previous section.

It follows from the definition of our term "stability of configuration" that the radial distribution of the field must be independent of the speed. It can be seen, from (2.3) and the discussion preceding (2.7), that the necessary condition for this is the mutual proportionality of both amplitude constants of the electric and of the magnetic field, *i. e.*,

$$\frac{d}{d\beta}\left(\frac{A_1}{A_2}\right)=\frac{d}{d\beta}\left(\frac{A_1'}{A_2'}\right)=0. \quad . \quad . \quad . \quad (5.1)$$

This condition will be, of course, always satisfied in the elliptically polarized Whirls (where $A_1=\pm A_2$). On the other hand, in the ellipsoidally polarized Whirls it will be satisfied if, and only if, the amplitude constants $A_e$ and $A_m$ of both component elliptically polarized wave-trains vary proportionally (by (2.12)).

Therefore, without loss of generality, we can limit our consideration of the amplitude constants to the case where the Whirl is elliptically polarized, *i. e.*, either electro-polarized or magnetopolarized.

The lines of force of the elliptically polarized vector ("Transversal" vector) lie in the $r\theta$ planes. During the process of transition from rest to axial movement, the whole field contracts $\beta$ times in the $x$ direction, *i. e.*, in the direction perpendicular to these lines of force.

It follows from the invariability of the radial distribution of e.m. field, that the shape of the lines of force of

* See paragraphs preceding formula (4.6).

the transversal vector does not alter as a result of uniform movement. Therefore *the flux of the transversal vector between two planes* $r\theta$, *invariably situated at the distance* $\frac{\Delta x}{\beta}$ *from each other, must remain constant*, since any infinitesimal variation of this flux *without change of the shape* of the lines of force would immediately provoke the formation of the lines of force of the longitudinal vector, and this process of formation of longitudinal lines of force would cause an opposition to the variation of the flux of the transversal vector.

The following simple experiment will illustrate this process:

Imagine a row of closely situated thin super-conducting rings, in which electric current is flowing. If we remove every alternate ring, the total current, as well as the relative distribution of this current, will remain unaltered, but if we alter the shape of the rings the current will be, generally speaking, altered.

The constancy of the flux of the transversal vector between any two planes at a distance $\frac{\Delta x}{\beta}$ is equivalent to the proportionality of the $r$ and $\theta$ components of the amplitude of the transversal vector, to $\beta$, which, in its turn, is equivalent to the proportionality to $\beta$, of the amplitude constant $A_e$ (see formula (2.9)).

Hence we can write

$$A_e = a\beta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.2)$$

where $a$ is a constant, which is independent of $\beta$. We shall call $a$ "invariant amplitude constant."

We shall now derive the expressions for the amplitudes of the e.m. field, trying possibly to segregate those parts of the expressions which are independent of the particular properties of a given Whirl and which, therefore, can be written in an invariant form.

There will be a complete similarity in the expressions for the transversal and for the longitudinal vector in both types of elliptically polarized Whirls, electropolarized and magnetopolarized. However, to fix our ideas, we shall assume in the following discussion that our Whirls are the electropolarized ones.

Denoting

$$\rho = q_0 r, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.4)$$

we can, by (2.9)–(2.11) and (5.2), taking into account the paragraph preceding formula (2.7), express the amplitudes of the e.m. field of the Whirl in the form.

$$\left.\begin{aligned} &E_x=-\beta a q_0{}^2 G_x, \quad E_r=i\beta a q_0{}^2 G_r, \quad E_\theta=\beta a q_0{}^2 G_\theta, \\ &B_x=-i\beta a q_0{}^2 H_x, \quad B_r=-\beta a q_0{}^2 H_r, \quad B_\theta=i\beta a q_0{}^2 H_\theta, \end{aligned}\right\} \quad (5.5)$$

where

$$\left.\begin{aligned} &G_x=0, \\ &G_r=\frac{2n}{\rho'}J_n(\rho'), \\ &G_\theta=J_{n+1}(\rho')-J_{n-1}(\rho'), \end{aligned}\right\} \quad . \quad . \quad . \quad (5.6)$$

and

$$\left.\begin{aligned} &H_x=\frac{2\zeta}{\sqrt{\zeta^2+1}}J_n(\rho'), \\ &H_r=\frac{1}{\sqrt{\zeta^2+1}}(J_{n+1}(\rho')-J_{n-1}(\rho')), \\ &H_\theta=\frac{2n}{\sqrt{\zeta^2+1}\,\rho'}J_n(\rho'), \end{aligned}\right\} \quad . \quad . \quad (5.7)$$

where

$$\rho'=\rho\sqrt{\chi^2+1}, \quad . \quad . \quad . \quad . \quad . \quad (5.7a)$$

and, by (4.10*a*) and (4.12),

$$\zeta=\frac{\sqrt{\chi^2+1}}{\sqrt{\beta^2\chi^2+\beta^2-1}}\,e^{-i\tan^{-1}\frac{\beta\chi}{\sqrt{\beta^2-1}}}, \quad . \quad . \quad (5.8)$$

and, by (4.10*b*) and (4.12),

$$\sqrt{\zeta^2+1}=\frac{\sqrt{\beta^2+\chi^2(\beta^2-1)}}{\sqrt{\beta^2\chi^2+\beta^2-1}}\,e^{i\left(\tan^{-1}\chi\frac{\sqrt{\beta^2-1}}{\beta}-\tan^{-1}\frac{\beta\chi}{\sqrt{\beta^2-1}}\right)}. \quad (5.9)$$

The introduction of the factor $q_0{}^2$ in (5.5) is, at the present stage, purely formal, for $q_0$ is, by definition, constant for a given Whirl (see 4.7)), but this will simplify our work in the future, when we shall deal with the e.m. energy of the Whirl.

We can see that, while G (which relate to the transversal vector) are independent of $\beta$, H (which relate to the longitudinal vector) depend on $\beta$ through $\zeta$.

Using the well-known asymptotic expansion formula of Bessel coefficients,

$$\left.\begin{aligned} J_n(z)&=P_n(z)\cos\left(z-\frac{2n+1}{4}\pi\right)+Q_n(z)\sin\left(z-\frac{2n+1}{4}\pi\right),\\ &\text{where}\\ P_n(z)&=\sqrt{\frac{2}{\pi z}}+o\{z^{-5/2}\},\\ Q_n(z)&=\frac{1-4n^2}{8z}\sqrt{\frac{2}{\pi z}}+o\{z^{-7/2}\}, \end{aligned}\right\} \quad \ldots \quad (5.10)$$

we can write the expressions (5.6) and (5.7), for

$$\rho'=\rho\sqrt{\chi^2+1}>>1,$$

in the following approximate form :

$$\left.\begin{aligned} G_x&\equiv 0,\\ G_r&=o\{\rho'^{-3/2}\},\\ G_\theta&=\frac{2\sqrt{\frac{2}{\pi}}\sin\left(\rho'-\frac{2n+1}{4}\pi\right)}{\rho'^{1/2}}+o\{\rho'^{-3/2}\}, \end{aligned}\right\} \quad (5.11)$$

and

$$\left.\begin{aligned} H_x&=\frac{2\sqrt{\frac{2}{\pi}}\zeta\cos\left(\rho'-\frac{2n+1}{4}\pi\right)}{\sqrt{\zeta^2+1}\ \rho'^{1/2}}+o\{\rho'^{-3/2}\},\\ H_r&=\frac{2\sqrt{\frac{2}{\pi}}\sin\left(\rho'-\frac{2n+1}{4}\pi\right)}{\sqrt{\zeta^2+1}\ \rho'^{1/2}}+o\{\rho'^{-3/2}\},\\ H_\theta&=o\{\rho'^{-3/2}\}. \end{aligned}\right\} \quad (5.12)$$

It could be seen from the consideration of terms of $o\{\rho'^{-3/2}\}$ of Bessel asymptotic expansion (formula (5.10)) that the relative error arising from neglecting this term is equal to $\frac{4n^2-1}{8\rho'}$. This is the most important error resulting from our approximation.

Considering the numerical tables of Bessel Functions, we can say, approximately, to give a general idea, that

we shall get the accuracy of about 2 per cent. if we start applying these formulæ after the second root of Bessel Functions, not counting the root at $\rho' = 0$.

By using two more terms of the asymptotic expansion (5.10), we can attain much greater accuracy, even at such a near point to the axis.

It is not difficult to see from (3.4), (4.11), (5.4), (5.5), (5.11), and (5.12) that the electromagnetic field of the Whirl is $2n$ pole in the angular ($\theta$) direction and forms cylindrical layers with an alternately changing polarity in the radial ($\rho$) direction, with an angular distance frequency approximately equal to $q_0\sqrt{\chi^2+1}$.

### 6. *The Energy and the Angular Momentum of the Whirl.*

By the " energy of the Whirl " we mean the energy of its e.m. field in the usual classical sense.

Denoting the total energy of the Whirl by $\mathbf{E}$, we can write

$$\mathbf{E} = \frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} (E_x^2+E_r^2+E_\theta^2+B_x^2+B_r^2+B_\theta^2)2\pi r\,dx\,dr, \quad . \; . \; . \quad (6.1)$$

Taking into account (3.4) and (4.5) and using mean square values with respect to time $t$, we can write (6.1) in the form

$$\mathbf{E} = \frac{1}{4}\int_{0}^{\infty}\int_{0}^{\infty} (|\mathrm{E}_x|^2+|\mathrm{E}_r|^2+|\mathrm{E}_\theta|^2+|\mathrm{B}_x|^2+|\mathrm{B}_r|^2+|\mathrm{B}_\theta|^2) \times e^{-2\phi_2} r\,dx\,dr, \quad (6.2)$$

where we have also substituted, with respect to $x$, the integration from zero to infinity for the integration from $-\infty$ to $\infty$, in view of the fact that the subintegral function is an even function of $x$. The symbols $|\mathrm{E}_x|$, etc., show in the usual way, that only the moduli of the amplitude expressions must be taken.

By (5.4), (5.5), (4.11), and (4.12), the formula (6.2) can be written in the form

$$\mathbf{E} = a^2\beta q_0 \mathbf{Q}, \quad . \; . \; . \; . \; . \; . \; . \quad (6.3)$$

where

$$\mathbf{Q} = \frac{1}{4}\int_{0}^{\infty}\int_{0}^{\infty} e^{-2\chi\xi}(|\mathrm{G}_x|^2+|\mathrm{G}_r|^2+|\mathrm{G}_\theta|^2 +|\mathrm{H}_x|^2+|\mathrm{H}_r|^2+|\mathrm{H}_\theta|^2)\rho\,d\xi\,d\rho, \quad (6.4)$$

which, as one can see at a glance from (5.6)–(5.9), does not depend on $q_0$ (*i. e.*, on the frequency, for $q_0=\Omega_0/c$) or on the invariant amplitude constant of the Whirl.

Denoting

$$\hbar_1=\frac{a^2}{c}\mathbf{Q}, \qquad \mathbf{E}_0=a^2q_0\mathbf{Q}=h_1\Omega_0,^* \quad . \quad . \quad (6.5)$$

we can, by (4.7) and (4.8), write (6.3) in the forms

$$\left.\begin{aligned} &\mathbf{E}=\hbar_1\Omega_1=|-n\Omega_{1x}|\hbar_1, \quad \text{by (3.6),}\\ \text{and} \qquad &\mathbf{E}=\beta\mathbf{E}_0=\beta\hbar_1\Omega_0, \end{aligned}\right\} \quad . \quad . \quad (6.6)$$

where $\mathbf{E}_0$ is the value of $\mathbf{E}$ at $\beta=1$.

Formulæ (6.6) show that, provided $\mathbf{Q}=$const., the well-known Einstein law will hold for the Whirls, *on the basis of Classical Electrodynamics*. However, even if $\mathbf{Q}=$const., $\hbar_1$ will *not* be the universal Planck constant, for it depends on the invariant amplitude constant $a$. It will only be a constant for a given Whirl.

On introducing into (6.4) the expressions from (5.6) and (5.7), and recalling that

$$\frac{2n}{z}\mathrm{J}_n(z)=\mathrm{J}_{n+1}(z)+\mathrm{J}_{n-1}(z), \quad . \quad . \quad . \quad (6.7)$$

we find the expression for $\mathbf{Q}$ which, after simple transformation, omitting, for brevity, the argument

$$\rho'=\rho\sqrt{\chi^2+1}$$

in the Bessel coefficients, we can write in the form

$$\mathbf{Q}=\frac{1}{2}\int_0^\infty\int_0^\infty e^{-2\chi\xi}\left[(\mathrm{J}_{n+1}^2+\mathrm{J}_{n-1}^2+2\mathrm{J}_n^2)+\frac{1}{|\zeta^2+1|}(\mathrm{J}_{n+1}^2+\mathrm{J}_{n-1}^2-2\mathrm{J}_n^2)\right]\rho\,d\xi\,d\rho. \quad (6.7a)$$

Hence, denoting

$$\left.\begin{aligned} \mathbf{C}_1&=\int_0^\infty\int_0^\infty e^{-2\chi\xi}(\mathrm{J}_{n+1}^2+\mathrm{J}_{n-1}^2+2\mathrm{J}_n^2)\rho'\,d\rho'\,d(2\chi\xi)\\ &=\int_0^\infty(\mathrm{J}_{n+1}^2+\mathrm{J}_{n-1}^2+2\mathrm{J}_n^2)\rho'\,d\rho',\\ \mathbf{C}_2&=\int_0^\infty\int_0^\infty e^{-2\chi\xi}(\mathrm{J}_{n+1}^2+\mathrm{J}_{n-1}^2-2\mathrm{J}_n^2)\rho'\,d\rho'\,d(2\chi\xi)\\ &=\int_0^\infty(\mathrm{J}_{n+1}^2+\mathrm{J}_{n-1}^2-2\mathrm{J}_n^2)\rho'\,d\rho', \end{aligned}\right\} \quad (6.7b)$$

* Strictly speaking, we should write here $|\Omega_0|$ instead of $\Omega_0$, as $\Omega_0$ can be negative. This, however, can be easily borne in mind.

we obtain

$$\mathbf{Q} = \frac{C_1}{4\chi(\chi^2+1)} + \frac{C_2}{4\chi(\chi^2+1)\,|\zeta^2+1|} \quad . \quad . \quad . \quad (6.8)$$

It can be easily seen that $C_1$ and $C_2$ are independent of the parameters of the Whirl.

The first term of the second member of (6.8) is invariant, while the second term depends on $\beta$ through $\zeta^2+1$.

To estimate the relative magnitude of $C_1$ and $C_2$, we shall use the approximate expression (5.10) for Bessel coefficients, neglecting the error arising from the application of this expression to the region where $\rho$ is not large enough for the use of the approximate expressions.

This is permissible, because when $\rho \to 0$, the sub-integral expressions in (6.7 *b*) do not become infinite.

We shall calculate, not the whole values of $C_1$ and $C_2$, but their values within a cylinder of a finite, but very large radius $\rho_2$. Consequently we shall take the integrals with respect to $\rho$, not from 0 to $\infty$, but from 0 to $\rho_2$. We shall denote these values by $C_{1\rho_2}$ and $C_{2\rho_2}$ respectively.

Then, on introducing into (6.7 *b*) the approximate expressions (5.10), we shall obtain

$$\left.\begin{aligned} C_{1\rho_2} &= \frac{4}{\pi}\rho_2'(1+o\{\rho_2'^{-1}\}), \\ C_{2\rho_2} &= (-)^n \frac{4}{\pi}\cos 2\rho_2' \,.\, (1+o\{\rho_2'^{-1}\}), \end{aligned}\right\} \quad . \quad (6.9)$$

where

$$\rho_2' = \rho_2\sqrt{\chi^2+1}.$$

Thus when $\rho_2 \to \infty$, $C_{1\rho_2}$ tends to infinity, while $C_{2\rho_2}$ remains finite.

We can now omit the suffix 2 from $\rho_2$. Denoting the value of $\mathbf{Q}$, calculated for a cylinder of radius $\rho$, by $\mathbf{Q}_\rho$, we can write by (6.3), (6.8), and (6.9),

$$\left.\begin{aligned} \mathbf{Q}_\rho &\underset{\rho \gg 1}{=} \frac{C_{1\rho}}{4\chi(\chi^2+1)}(1+o\{\rho'^{-1}\}), \\ \mathbf{Q} &= \lim_{\rho \to \infty} \mathbf{Q}_\rho. \end{aligned}\right\} \quad . \quad . \quad (6.10)$$

Denoting by $\hbar_{1\rho}$ the value of $\hbar_1$ within a cylinder of radius $\rho$, we find by (6.8), (6.10), and (5.7*a*),

$$\left.\begin{aligned} \hbar_{1\rho} &= \frac{a^2\rho}{\pi c\chi\sqrt{\chi^2+1}}(1+o\{(\rho\sqrt{\chi^2+1})^{-1}\}), \\ \hbar_1 &= \lim_{\rho\to\infty}\hbar_{1\rho}. \end{aligned}\right\} \quad (6.11)$$

Denoting by $\mathsf{E}_\rho$ the energy of the Whirl within the radius $\rho$, we obtain by (6.6) and (6.11)

$$\left.\begin{aligned} \mathsf{E}_\rho &= \Omega_1\hbar_{1\rho} \cong \frac{\Omega_0\beta a^2\rho}{\pi c\chi\sqrt{\chi^2+1}}(1+o\{(\rho\sqrt{\chi^2+1})^{-1}\}), \\ \mathsf{E} &= \lim_{\rho\to\infty}\mathsf{E}_\rho. \end{aligned}\right\} \quad (6.12)$$

We shall now interpret the physical meaning of $\hbar_1$.

If we have a system, the energy of which is a product of an angular velocity and of a second factor, then, according to Classical Mechanics, the second factor, whatever be the mechanism of the system, equals half the angular momentum of the system. Since we found, on the basis of Classical Electrodynamics, that the energy of the Whirl is equal to $-\Omega_{1z}\, n\hbar_1$ (by (6.6)), we must consider $-2n\hbar_1$, not by analogy, but in a true sense, as the angular momentum of the Whirl.

This angular momentum remains invariant, which, indeed, should be expected, for we regarded the Whirl as a self-contained system.

## 7. *Incomplete Whirls.*

We can see from (5.4) and (6.12) that, if the magnitudes of electric and magnetic vectors are finite, $\mathsf{E}$ is infinite (and hence the above described Whirl is physically impossible), unless $\chi=\infty$, or, in other words, unless the Whirl is infinitely thin (by (4.11) and (3 4)).

It is hardly conceivable that such an infinitely wide and thin Whirl could be formed simultaneously all over space. It must start from limited dimensions and then spread.

We shall try now to form a general picture of this spreading.

We shall assume that, within a large but limited radius $\rho={}^0\rho_2>>1$, an e.m. field has been somehow formed,

which, at the given moment (*i. e.*, when the formation has been just completed) satisfies, within the limited radius ${}^0\rho_2$, all the relationships appropriate to a Whirl *.

Owing to the exponential nature of the axial distribution of the energy, it is possible to satisfy the equations of the Whirl, in the axial direction, within a sufficient length to make the total energy, which, according to these equations, is situated beyond this length, very small in comparison with the energy within it, and, correspondingly, make the energy density at the boundary so negligible that the maximum possible energy-flux, through this boundary in the $x$ direction practically would not influence the axial distribution of the energy within this boundary.

We shall assume, therefore, that during the spreading of the incomplete Whirl

$$\chi = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.1)$$

We shall denote by $\rho_2$ the value of $\rho$ for the outside boundary of the incomplete Whirl.

Since an incomplete Whirl is a self-contained system, the relationships (6.11) and (6.12), which express the laws of conservation of energy and angular momentum, must be exactly observed for $\rho=\rho_2$ (outside boundary). In other words,

$$\left.\begin{aligned} \mathsf{E}_{\rho_2} &= \mathsf{E}_{0\rho_2}, \\ \hbar_{\rho_2} &= \hbar_{0\rho_2}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.2)$$

whence, by (6.11) and (7.1),

$$\rho_2 = {}^0\rho_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.2a)$$

(it will be recalled that ${}^0\rho_2$ is the value of $\rho_2$ at the commencement of spreading).

We shall assume that the incomplete Whirl conserves the form of its structure during its spreading.

Then we can write

$$\rho = {}^0\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.3)$$

where ${}^0\rho$ is the value of $\rho$ at the commencement of spreading. Formula (7.3) really means that the expression

* It will be recalled that $\rho$ measures the radius in abstract units depending on the parameters of the Whirl ($\rho = q_0 r$) and not in units of length.

for the e.m. field during the spreading of an incomplete Whirl is invariant in terms of $\rho$, which, of course, follows from our assumption of the conservation of form of structure.

It follows, further, from this assumption, by (7.3), (6.11), (6.12), and (7.1), that the energy and the angular momentum within an annulus, limited by two cylindrical surfaces $\rho$ and $\rho+\Delta\rho$, remain constant (within the accuracy of our formulæ, *i.e.*, to the relative order $\rho^{-1}$).

This makes our assumption plausible, for it means the absence of any appreciable forces of interaction between different parts of the spreading incomplete Whirl, which would tend to interfere with the assumed structure *.

We shall now investigate the variation, during the spreading, of other data relating to the incomplete Whirl.

We shall denote by a suffix $^0$ at the top on the left, the value of any quantity at the commencement of spreading (corresponding to ${}^0\rho$ for $\rho$).

By (7.3), (6.12), and (7.1), taking into account that $\mathsf{E}_\rho$ must, according to our assumption, remain constant,

$$\beta\Omega_0={}^0\beta\,{}^0\Omega_0=\text{const.}, \quad . \quad . \quad . \quad . \quad (7.4)$$

or by (4.8),

$$\Omega_1={}^0\Omega_1=\text{const.} \quad . \quad . \quad . \quad . \quad . \quad (7.5)$$

Denoting the radii corresponding to $\rho$ and $\rho_2$ by $r$ and $r_2$ respectively, we can, taking into account (4.7) and (5.4), write (7.2$a$) and (7.3) in the form

$$\left.\begin{aligned}\Omega_0 r_2&={}^0\Omega_0\,{}^0r_2,\\ \Omega_0 r&={}^0\Omega_0\,{}^0r,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (7.6)$$

whence, by (7.4),

$$\frac{\beta}{{}^0\beta}=\frac{{}^0\Omega_0}{\Omega_0}=\frac{r_2}{{}^0r_2}=\frac{r}{{}^0r}. \quad . \quad . \quad . \quad . \quad (7.7)$$

Assume now that the outer radius increases during the spreading with a speed $c_3$, and that the spreading has begun at the time $t=0$. Then

$$r_2={}^0r_2+c_3t. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.8)$$

* It is quite legitimate to test the possibility of a dynamical process by assuming it, and then testing whether this process would cause any forces of interaction between different parts of the system, tending to interfere with this process. If this be the case then, obviously, the process cannot take place naturally, if not—it can.

By (7.7) and (7.8),

$$\left.\begin{aligned} r &= {}^0r\left(1+\frac{c_3 t}{{}^0r_2}\right) = {}^0r\left(1+\frac{q_0 c_3 t}{\rho_2}\right), \\ \beta &= {}^0\beta\left(1+\frac{c_3 t}{{}^0r_2}\right) = {}^0\beta\left(1+\frac{q_0 c_3 t}{\rho_2}\right). \end{aligned}\right\} \quad . \quad . \quad (7.9)$$

This shows that the incomplete Whirl must continually widen, the slower the greater was its initial radius, and that the coefficient $\beta$ must increase in the same proportion.

This increase of $\beta$ means that the speed of its translatory axial movement will approach the speed of light (formula (4.4)), and the wave-length $\lambda$ will approach that of a light wave (by (4.10)) while its frequency will remain constant (formula (7.5)).

When $\beta \to \infty$ the coefficient $\zeta \sim \beta^{-1}$ (formula (4.10 $a$), and hence $\frac{B_x}{B_r} \to 0$ (formulæ (5.5)–(5.7)). In other words, both vectors of the e.m. waves forming the incomplete Whirl become elliptically, and not ellipsoidally, polarized.

When $\rho >> 1$ only two vectors will practically remain: $E_\theta$ and $B_r$.

It can be noted that the incomplete Whirl transforms itself into a system which has the properties of the light quantum (photon).

The maximum possible value of the speed $c_3$ of spreading of the outer boundary of the incomplete Whirl (formula (7.8)) is, of course, the speed of light $c$, but it seems that it should slow down with the increase of $\rho_2$, for with this the incomplete Whirl will approach more to the complete one, and the electromagnetic vectors will become more mutually balanced in the process of rotation and axial propagation (as they indeed are in the complete Whirl) and, therefore, the speed of lateral spreading, which is required to complete this balance, will be smaller.

However, we shall not consider here the question on the value of $c_3$ any further.

## 8. *Compound Whirls.*

We shall call "Compound Whirl" a wave-train consisting of an infinite number of infinitesimal co-axial Whirls.

We shall limit our consideration to the case where not only the axis of symmetry of all the component Whirls is common, but also they have a common central plane. We shall assume that all the component Whirls have a *common real phase* $\phi_1$ (which by (4.11) also involves a common $\Omega_0$ and $\beta$), while, on the other hand, the "imaginary" phases $\phi_2$ vary (which by (4.11) and (4.12) means that $\chi$ must also vary).

To distinguish the compound Whirls from those which we considered in the previous section (*i. e.*, Whirls with a definite value of $\chi$) we shall call the latter "simple Whirls." Thus a compound Whirl consists of an infinite number of infinitesimal simple Whirls.

Assume that the infinitesimal invariant amplitude constant of an infinitesimal simple Whirl is $da$, while the parameter $\chi$ lies between $\chi-\frac{d\chi}{2}$ and $\chi+\frac{d\chi}{2}$, and denote

$$A\alpha=\frac{da}{d\chi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.1)$$

where $A$ is a constant, while $\alpha$ is a variable. The expressions for the components $dE_x$, etc., of the e.m. field of a component simple Whirl will be obtained from (3.4) and (5.5) on substituting $dE_x$, etc., for $E_x$, etc., and $da$, drawn from (8.1), for $a$.

$$\left.\begin{aligned} dE_x &= -\beta A\alpha q_0^2 G_x\, d\chi . e^{-\phi_2+i\phi_1}, \\ dE_r &= i\beta \;,, G_r \;,, \\ dE_\theta &= \beta \;,, G_\theta \;,, \\ dB_x &= -i\beta \;,, H_x \;,, \\ dB_r &= -\beta \;,, H_r \;,, \\ dB_\theta &= i\beta \;,, H_\theta \;,, \end{aligned}\right\} \quad . \quad . \quad . \quad (8.2)$$

We shall generally assume that all the values of $\chi$ in our compound Whirl represent one or several continuous sets and that in each set $\chi$ is enclosed within an interval $\chi_1 \leq \chi \leq \chi_2$, within which $\alpha$ and $\frac{d\alpha}{d\chi}$ are finite and single valued. Without loss of generality we can limit ourselves to the case where there is only one continuous set of $\chi$, for in the general case the compound Whirl can be

represented as a resultant of several compound Whirls with one continuous set of $\chi$.

Moreover, without loss of generality, we can assume that $\chi_2 = \infty$, for

$$\int_{\chi_1}^{\chi_2} = \int_{\chi_1}^{\infty} - \int_{\chi_2}^{\infty}.$$

We can, therefore, obtain the expressions for the vector components of a compound Whirl by integrating (8.2) from $\chi_1$ to $\infty$.

The result can be written as follows:

$$\left.\begin{array}{ll} E_x = \underset{\theta}{\mathrm{E}_x} e^{i\phi_1}, & B_x = \underset{\theta}{\mathrm{B}_x} e^{i\phi_1}, \\ \underset{r}{} & \underset{r}{} \end{array}\right\} \quad . \quad . \quad . \quad (8.3)$$

where

$$\left.\begin{array}{ll} \mathrm{E}_x = -\beta A q_0^2 \mathrm{L}_x, & \mathrm{B}_x = -i\beta A q_0^2 \mathrm{M}_x, \\ \mathrm{E}_r = i\beta A q_0^2 \mathrm{L}_r, & \mathrm{B}_r = -\beta A q_0^2 \mathrm{M}_r, \\ \mathrm{E}_\theta = \beta A q_0^2 \mathrm{L}_\theta, & \mathrm{B}_\theta = i\beta A q_0^2 \mathrm{M}_\theta, \end{array}\right\} \quad . \quad (8.4)$$

where

$$\left.\begin{array}{ll} \mathrm{L}_x \equiv 0, & \mathrm{M}_x = \int_{\chi_1}^{\infty} \alpha e^{-\chi\xi} \mathrm{H}_x d\chi, \\ \mathrm{L}_r = \int_{\chi_1}^{\infty} \alpha e^{-\chi\xi} \mathrm{G}_r d\chi, & \mathrm{M}_r = \int_{\chi_2}^{\infty} \alpha e^{-\chi\xi} \mathrm{H}_r d\chi, \\ \mathrm{L}_\theta = \int_{\chi_1}^{\infty} \alpha e^{-\chi\xi} \mathrm{G}_\theta d\chi, & \mathrm{M}_\theta = \int_{\chi_1}^{\infty} \alpha e^{-\chi\xi} \mathrm{H}_\theta d\chi. \end{array}\right\} \quad (8.5)$$

Further, without any loss of generality (see (8.1)), we can put

$$\alpha = 1 \quad \text{when} \quad \chi = \chi_1. \quad . \quad . \quad . \quad . \quad (8.6)$$

If we denote by $\kappa_{21}$ the value of $\kappa_2$ (see formula (3.1)) corresponding to $\chi = \chi_1$, we shall have, by the third formula (4.12),

$$\chi_1 = \frac{\kappa_{21}}{\beta q_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.6a)$$

The functions L and M will only depend on the constant $\chi_1$, and on the form of the function $\alpha(\chi)$.

By introducing into (8.5) the values from (5.6)–(5.9), we can obtain the complete expressions of L and M.

To get the approximate expressions for L and M, we can use the approximate expressions (5.11) and (5.12) of G and H, instead of the exact expressions (5.6) and (5.7).

We shall limit ourselves to the calculation of L and M in the case $\rho >> 1$, to the order of $\rho^{-3/2}$ and shall use, therefore, the latter expressions. On doing this we shall obtain

$$\left.\begin{aligned} &L_x \equiv 0, \\ &L_r = o\{L_\theta \rho^{-1}\}. \\ &L_\theta = 2\sqrt{\frac{2}{\pi}} \int_{\chi_1}^{\infty} \alpha e^{-\chi\xi} \frac{\sin\left(\rho\sqrt{\chi^2+1} - \frac{2n+1}{4}\pi\right)}{(\rho\sqrt{\chi^2+1})^{\frac{1}{2}}} d\chi. \end{aligned}\right\} \quad (8.7)$$

$$\left.\begin{aligned} &M_x = 2\sqrt{\frac{2}{\pi}} \int_{\chi_1}^{\infty} \frac{\zeta \alpha e^{-\chi\xi}}{\sqrt{\zeta^2+1}} \frac{\cos\left(\rho\sqrt{\chi^2+1} - \frac{2n+1}{4}\pi\right)}{(\rho\sqrt{\chi^2+1})^{\frac{1}{2}}} d\chi, \\ &M_r = 2\sqrt{\frac{2}{\pi}} \int_{\chi_1}^{\infty} \frac{\alpha e^{-\chi\xi}}{\sqrt{\zeta^2+1}} \frac{\sin\left(\rho\sqrt{\chi^2+1} - \frac{2n+1}{4}\pi\right)}{(\rho\sqrt{\chi^2+1})^{\frac{1}{2}}} d\chi, \\ &M_\theta = o\{M_r \rho^{-1}\}. \end{aligned}\right\}$$

. . . (8.8)

where the $\zeta$ and $\zeta^2+1$ must be drawn from (5.8) and (5.9).

To calculate this integral to the order $\rho^{-3/2}$, we shall use the formula (2.2 ap.) from the Appendix to Part I., according to which

$$\int_{\chi_1}^{\infty} f(\chi) e^{-\chi\xi} \begin{matrix}\cos \\ \sin\end{matrix}\left(\rho\sqrt{\chi^2+1} - \frac{2n+1}{4}\pi\right) d\chi$$

$$= \frac{f(\chi_1) e^{-\chi_1}}{\sqrt{\xi^2+\rho^2}} \begin{matrix}\cos \\ \sin\end{matrix}\left(\rho\sqrt{{\chi_1}^2+1} - \frac{2n+1}{4}\pi - \tan^{-1}\frac{\rho}{\xi}\right)$$

$$+ o\{(\xi^2+\rho^2)^{-1}\}, \quad . \quad . \quad . \quad (8.9)$$

whence, taking into account (8.6), we obtain from (8.7) and (8.8),

$$\left.\begin{aligned} &L_x \equiv 0, \\ &L_r = o\{L_\theta (\rho\sqrt{{\chi_1}^2+1})^{-1}\}. \\ &L_\theta = \frac{2\sqrt{\frac{2}{\pi}}\, e^{-\chi_1\xi} \sin\left(\Psi - \frac{2n+1}{4}\pi\right)}{(\rho\sqrt{{\chi_1}^2+1})^{\frac{1}{2}} \sqrt{\xi^2+\rho^2}}. \end{aligned}\right\} \quad . \quad (8.10)$$

$$\left.\begin{aligned} M_x &= \frac{\zeta_1}{\sqrt{\zeta_1^2+1}} \frac{2\sqrt{\frac{2}{\pi}}\, e^{-\chi_1\xi}\cos\left(\Psi-\frac{2n+1}{4}\pi\right)}{(\rho\sqrt{\chi_1^2+1})^{\frac{1}{2}}\sqrt{\xi^2+\rho^2}}, \\ M_r &= \frac{1}{\sqrt{\zeta_1^2+1}} \frac{2\sqrt{\frac{2}{\pi}}\, e^{-\chi_1\xi}\sin\left(\Psi-\frac{2n+1}{4}\pi\right)}{(\rho\sqrt{\chi_1^2+1})^{\frac{1}{2}}\sqrt{\xi^2+\rho^2}}, \\ M_\theta &= o\{M_r(\rho\sqrt{\chi_1^2+1})^{-1}\}. \end{aligned}\right\} \quad (8.10\,a)$$

where

$$\Psi=\rho\sqrt{\chi_1^2+1}-\tan^{-1}\frac{\rho}{\xi}, \quad . \quad . \quad . \quad (8.11)$$

and $\zeta_1$, is the value of $\zeta$ at $\chi=\chi_1$.

By (5.8) and (5.9),

$$\left.\begin{aligned} \zeta_1 &= \frac{\sqrt{\chi_1^2+1}}{\sqrt{\beta^2\chi_1^2+\beta^2-1}} e^{-i\tan^{-1}\frac{\beta\chi_1}{\sqrt{\beta^2-1}}} = |\zeta_1|\, e^{i(\delta+\gamma)}, \\ \sqrt{\zeta_1^2+1} &= \frac{\sqrt{\beta^2+\chi_1^2(\beta^2-1)}}{\sqrt{\beta^2\chi_1^2+\beta^2-1}} e^{i\left(\tan^{-1}\frac{\chi_1\sqrt{\beta^2-1}}{\beta}-\frac{\tan^{-1}\beta\chi_1}{\sqrt{\beta^2-1}}\right)} \\ &= |\sqrt{\zeta_1^2+1}|\, e^{i\gamma}, \end{aligned}\right\}$$

. . . (8.12)

where $|\zeta_1|$ and $\delta+\gamma$, and $|\sqrt{\zeta_1^2+1}$ and $\gamma$ are moduli and arguments of $\zeta_1$, and $\sqrt{\zeta_1^2+1}$ respectively.

It should be noted that the second equation of (8.10) and the third equation of (8.10 *a*) cannot be, strictly speaking, simply written by analogy with the corresponding equations of (5.11) and (5.12), but it would not be difficult to do this rigorously by putting the corresponding equations in (5.11) and (5.12) fully, and then integrating.

Using more terms of the asymptotic expansion in (5.10) and correspondingly more terms in formula (2.2 ap.) the values of $L_x$, etc., for $\rho\sqrt{\chi_1^2+1} > > 1$ can be calculated, with a close approximation.

It follows from the observation made at the end of section 5 about the relationship between the accuracy of approximate formulæ (5.11) and (5.12), that the maximum relative error given by the formulæ (8.10) and (8.11) is about $\frac{4n^2-1}{8\rho\sqrt{\chi_1^2+1}}$, and that the minimum accuracy to

about 2 per cent. will be obtained if we apply these formulæ after the second root of $J_n(\rho\sqrt{\chi_1^2+1})$.

The calculation of $L_x$, etc., for smaller values can be performed without great difficulty, but we shall not do it here, for this is not required for our immediate purpose.

We can see from (8.10) and (8.10*a*) that for $\rho\sqrt{\chi_1^2+1} >> 1$, and within the relative accuracy to the order $(\rho\sqrt{\chi_1^2+1})^{-1}$, the distribution of the field does not depend on the form of the function $\alpha(\chi)$, and depends only on $\chi_1$, so that the same formulæ are applicable to any compound Whirl with a given $\chi_1$. The only thing which is required, is that $\alpha(\chi)$ be continuous within $\chi_1 \leq \chi \leq \infty$. In the case of a discontinuity in $\alpha(\chi)$, the Whirl must be considered as a resultant of several (as mentioned above) and then, of course, the value of $\chi$ at each point of discontinuity will reflect on our formulæ.

The essential difference between the simple Whirl and the compound one is that while in the former the field intensity at $\rho >> 1$ is of the order $\rho^{-1/2}$, in the latter it is of the order of $[\rho(\rho^2+\xi^2)]^{-1/2}$.

The general character of phase distribution in a compound Whirl is the same as in a simple one.

If we try to visualize the structure of the electromagnetic field expressed by (8.3), (8.4), (8.10), and (8.10*a*) we can see that this field is $2n$ pole in the angular ($\theta$) direction, and that, not too near to the centre, it forms layers alternately changing their polarity in the radial ($\rho$) direction with the angular distance frequency $q_0\sqrt{\chi_1^2+1}$. In the axial direction the e.m. field is exponentially periodic if the Whirl has a translatory motion along the axis of symmetry and purely exponential if the Whirl is at rest (by (8.3) and (4.11)).

### 9. *The Energy of Compound Whirls.*

The formulæ (6.1) and (6.2), which were used as a starting point for determining the energy of simple Whirls, are, of course, quite general, and can be equally used as a starting point for determining the energy of compound Whirls. Following exactly the same method as we did in section 6, we can write the expression for the energy of a compound Whirl in the form

$$\mathsf{E} = A^2\beta q_0 \mathsf{Q}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.1)$$

where

$$\mathbf{Q} = \frac{1}{4}\int_0^\infty \int_0^\infty (|\mathrm{L}_x|^2 + |\mathrm{L}_r|^2 + |\mathrm{L}_\theta|^2 + |\mathrm{M}_x|^2 + |\mathrm{M}_r|^2 + |\mathrm{M}_\theta|^2) \times \rho\, d\xi\, d\rho = \frac{1}{16\pi}\int \frac{d\mathbf{Q}}{d\tau} d\tau, \quad (9.2)$$

where $d\tau$ denotes an element of volume in the units corresponding to $\rho$ and $\xi$.

Apart from the difference in notation, formulæ (9.1) and (9.2) for compound Whirls will differ from the corresponding formulæ (6.3) and (6.4) for simple Whirls by the absence in the former of an explicit exponential factor $e^{-2\chi\xi}$ in the subintegral expression, for in compound Whirls a continuous set of factors of this nature is included in the amplitude (see previous section).

Following exactly the same way which we used in the case of simple Whirls, we shall denote

$$\left.\begin{aligned} &\frac{A^2}{c}\mathbf{Q} = \hbar_1, \\ \text{and} \quad &A^2 q_0 \mathbf{Q} = \mathbf{E}_0 = \hbar_1 \Omega_0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9.3)$$

and write (9.1) in the form

$$\left.\begin{aligned} &\mathbf{E} = \hbar_1 \Omega_1 = \beta \hbar_1 \Omega_0 = |1 - \Omega_{12} n|\, \hbar_1 \\ \text{and} \quad &\mathbf{E} = \beta \mathbf{E}_0 = \beta \hbar_1 \Omega_0, \end{aligned}\right\}, \quad . \quad . \quad (9.4)$$

which is the exact copy of the corresponding formulæ (6.6).

Since $\hbar_1$ is a quantity which can be, in general, different for different Whirls, there is no reason to use a distinct notation for it in the case of compound Whirls.

Again, formulæ (9.4) show that in the compound Whirl the well-known Einstein laws, which they express formally, will be observed if $\mathbf{Q}$ is invariant.

This question we shall now consider.

It has been stated in the previous section, that, when $\rho$ is greater than a certain value $\rho_1$, which depends on the accuracy of our calculation, the functions L and M can be determined to the relative order of accuracy $(\rho\sqrt{\chi_1^2+1})^{-1}$, by means of approximate formulæ (8.10)–(8.11) which were obtained by using, in the expressions for L and M, only the first term of the asymptotic expansion in the form (5.10) of Bessel functions, on which L

and M depend, instead of using the whole expression for these Bessel functions. It was shown that for $\rho > \rho_1$, and within this accuracy, L and M do not depend on the form of $\alpha(\chi)$, but only on $\chi_1$, *i. e.*, on the lower limit of $\chi$ in the compound Whirl ; it did not, however, follow from our discussion that the same is true when $\rho < \rho_1$.

Therefore, if we investigated now the energy distribution within the radius corresponding to $\rho = \rho_1$ we should not only have considerably increased our work, but would also lose, to a considerable extent, a generality combined with definiteness, which is possible if we limit ourselves to the domain corresponding to $\rho > \rho_1$.

As has been stated in the previous section, the value of $\rho_1$, corresponding to a reasonable accuracy, is not big. For instance, to a relative accuracy of about 2 per cent. in determination of L and M will correspond the second root of $J_n(\rho\sqrt{\chi_1^2+1})$, not counting the root at $\rho=0$. We shall see below (section 20) that, in the actual application of our results to the theory of elementary particles, this value of $\rho$ will correspond to an extremely small distance from the axis and will not at all affect the questions with which we shall deal. Therefore, the taking into account of the region $\rho < \rho_1$ would not justify the loss of generality or definiteness of our further discussion.

For this reason we shall ignore the energy within $\rho < \rho_1$ in the case of compound Whirls as we did in the case of simple ones (section 6). There will be, however, a difference in the method of excluding this region.

In the case of simple Whirls we did this by applying the approximate formulæ from zero to infinity, because the energy contained, according to the approximate formulæ, within the space corresponding to $\rho < \rho_1$ was very small. If, however, we applied the same method in the case of compound Whirls, the energy within the space corresponding to $\rho < \rho_1$ would appear to be infinite, which, of course, would be inconsistent with the results which would be obtained if the accurate formulæ were used.

We must, therefore, definitely segregate the value of Q within the radius corresponding to $\rho = \rho_1$ from that corresponding to $\rho > \rho_1$.

We shall put, therefore,

$$Q = Q_1 + Q_2, \quad . \quad . \quad . \quad . \quad . \quad (9.5)$$

where

$$\left.\begin{aligned}\mathbf{Q}_1&=\frac{1}{4}\int_0^{\infty}\int_0^{\rho_1}(|\mathrm{L}_x|^2+|\mathrm{L}_r|^2+|\mathrm{L}_\theta|^2+|\mathrm{M}_x{}^2|+|\mathrm{M}_r{}^2|+|\mathrm{M}_\theta|^2)\\&\qquad\times\rho\,d\xi\,d\rho,\\\mathbf{Q}_2&=\frac{1}{4}\int_0^{\infty}\int_{\rho_1}^{\infty}(|\mathrm{L}_x|^2+|\mathrm{L}_r{}^2|+|\mathrm{L}_\theta|^2+|\mathrm{M}_x|^2+|\mathrm{M}_r|^2+|\mathrm{M}_\theta|^2)\\&\qquad\times\rho\,d\xi\,d\rho,\end{aligned}\right\}\quad\ldots\quad(9.6)$$

and assume that $\rho_1$ is chosen as small as possible, but so as to satisfy the condition that for $\rho>\rho_1$, *i. e.*, for the calculation of $\mathbf{Q}_2$, the approximate expressions (8.10)–(8.11) of L and M can be used for a given degree of accuracy.

It is not difficult to see from (5.6), (5.7), (5.11), (5.12), and (8.5) that $\mathbf{Q}_1$ is finite, for the subintegral expression is always finite and falls, according to an exponential law, in the axial direction, and the integration is done over a limited area in a radial direction.

On the other hand, the question of finiteness or otherwise of $\mathbf{Q}_2$ must be considered.

By (8.10)–(8.12) and the second equation (9.6), we obtain, after simple transformations,

$$\begin{aligned}\mathbf{Q}_2&=\frac{1}{4}\int_0^{\infty}\int_{\rho_1}^{\infty}(|\mathrm{L}_\theta|^2+|\mathrm{M}_z|^2+|\mathrm{M}_r|^2)\rho\,d\xi\,d\rho\\&\qquad\times(1+o\{(\rho\sqrt{\chi_1{}^2+1})^{-1}\})\\&\simeq\frac{2}{\pi}\int_0^{\infty}\int_{\rho_1}^{\infty}\frac{e^{-2\chi_1\xi}}{\sqrt{\chi_1{}^2+1}\,(\xi^2+\rho^2)}d\xi\,d\rho\\&=\frac{2}{\pi\sqrt{\chi_1{}^2+1}}\int_0^{\infty}\frac{e^{-2\chi_1\xi}}{\xi}\left(\frac{\pi}{2}-\tan^{-1}\frac{\rho_1}{\xi}\right)d\xi.\quad\ldots\quad(9.7)\end{aligned}$$

Thus $\mathbf{Q}_2$ is independent of $\beta$.

Further, we can see that

$$\lim_{\xi=0}\frac{\frac{\pi}{2}-\tan^{-1}\frac{\rho_1}{\xi}}{\xi}=\frac{1}{\rho_1{}^2}.\quad\ldots\quad(9.8)$$

Thus the subintegral expression in (9.7) remains always finite. Besides, when $\xi\to\infty$ it obviously tends to become of a smaller order than $\xi^{-1}$.

Therefore $\mathbf{Q}_2$ is finite, and hence $\mathbf{Q}$ is finite (formula

(9.5)), and consequently, the energy of the compound Whirl is also finite (formula (9.1)). In this respect our compound Whirl differs essentially from the simple one, for the latter has an infinite energy. Therefore, while a simple Whirl can exist only in an "incomplete" state of continuous expansion and acceleration (section 7), a compound one can physically exist in a complete state.

While $\mathbf{Q}_2$ is independent of $\beta$, $\mathbf{Q}_1$ must, generally speaking, depend on $\beta$ through $\zeta$ (formulæ (9.6), (8.5), and (5.6)–(5.9)). This influence is not very explicit, because $\beta$ enters into the subintegral expressions of integrals which, in their turn, enter in the formulæ as squares, and for which we are unable to find a general expression. Since we decided not to include the numerical investigation of particular cases of $\alpha(\chi)$ in the present work, we cannot attempt to investigate here the question of the influence of $\beta$ on $\mathbf{Q}$ fully, and must limit ourselves to a few general remarks.

First of all, in the subintegral expressions in (8.5), the Bessel functions entering into G and H are functions of $\rho\sqrt{\chi^2+1}$ (by (5.6)–(5.9)), and the integration is performed with respect to $\chi$ from $\chi_1$ to infinity. Therefore, for the bulk of the interval of integration the approximate formulæ (8.7) and (8.8) are applicable, and it appears that the region of their applicability extends to a space within the radius corresponding to the first maximum of $J_n(\rho\sqrt{\chi_1^2+1})$, so that the portion of the total energy which may depend on $\beta$ appears to be comparatively very insignificant.

There is, however, apart from this "reasonable guess" in estimating $\mathbf{Q}_1$, another important consideration which makes us believe that if we were able to determine $\mathbf{Q}_1$ accurately, we should probably find it to be independent of $\beta$.

As we have said before (at the end of section 6), $-2n\hbar_1$ represents the angular momentum of the Whirl. Variations of $\beta$ result from the variations of speed of *translatory* motion, and this, according to Classical Mechanics, cannot have any influence on the angular momentum.

We can take it, therefore, that $\mathbf{Q}$ is independent of $\beta$.

Hence the equations (9.4) exactly correspond to the well-known Einstein law. But, again, $\hbar_1$ is *not* the Planck universal constant, but only a constant of a given Whirl.

By substituting spherical coordinates for the cylindrical ones, according to the formulæ

$$\left.\begin{aligned} \rho &= \mathrm{R}\sin\varphi, \quad \xi = \mathrm{R}\cos\varphi, \\ \mathrm{R}^2 &= \rho^2+\xi^2 = q_0{}^2(r^2+\beta^2x^2), \end{aligned}\right\} \quad . \quad . \quad (9.10)$$

the integral (9.7) can be presented in the form

$$\mathbf{Q}_2 \cong \frac{2}{\pi\sqrt{\chi_1{}^2+1}}\int_{\rho_1}^{\infty}\int_0^{\tan^{-1}\frac{\rho_1}{\mathrm{R}}} \frac{e^{-2\chi_1\mathrm{R}\cos\varphi}}{\mathrm{R}}\, d\mathrm{R}\, d\varphi, \quad (9.11)$$

where the distribution of energy within the compound Whirl will appear in a clearer way. It may well be seen from (9.11) that the total energy in an ellipsoidal layer is approximately in inverse proportion to the major axis of this layer. This law, of course, breaks down near to the axis where, according to our convention, we leave the region of $\mathbf{Q}_2$ and enter into the region of $\mathbf{Q}_1$.

When $\beta=1$ these ellipsoidal layers become spherical. In any case the ellipsoids can be replaced by spheres, using the invariant coordinates $\rho$ and $\xi$.

Near the equator the energies in each zone, containing the same angle $\varphi$, are equal, so that the energy density grows towards the poles. This tendency breaks down the sooner, the greater is $\beta\chi_1 q_0$. If $\chi_1$ is small in comparison with $(\beta q_0)^{-1}$, the energy density reaches its maximum near the poles.

When $\chi_1=0$, we, strictly speaking, leave the domain of the Whirls, for we include as an infinitesimal component a rotating cylindrical wave-train which is not exponentially periodic in $x$, but simply periodic. In this case the total energy of system becomes infinite, and, therefore, such a system can only exist in an "incomplete" state, continually spreading as in the case of a simple Whirl.

## 10. *Lateral Motion of the Whirl. Addition of Translatory Speeds.*

Let us consider a system of e.m. oscillations in free space, which are exponentially periodic in time. Such a system can be represented in cylindrical coordinates in the form

$$\left.\begin{aligned} \underset{\theta}{E_x} &= \mathrm{F}_{1x}(x, r, \theta)e^{i\Omega t}, \quad \underset{\theta}{B_x} = \mathrm{F}_{2x}(x, r, \theta)e^{i\Omega t}, \end{aligned}\right\} \quad (10.1)$$

where, generally speaking, $\Omega$ is a complex constant, and F can also be a complex quantity.

Since the field has no discontinuities, we can always represent the functions F in the form

$$\mathrm{F} = \sum_q F_q(r, \theta) e^{i\kappa_q z}, \quad . \quad . \quad . \quad . \quad (10.2)$$

where $\kappa_q$ is a constant. Formula (10.2) must include the case where the sum, or parts of the sum, consist of an infinite number of infinitesimal terms which can be represented in the form

$$dF_q = F_q'(\kappa_q, r, \theta) d\kappa_q,$$

and the whole part can be written in the form of

$$\int F_q'(\kappa_q, r, \theta) e^{i\kappa_q z} d\kappa_q.$$

However, for our purpose we need not consider this case of a sum in the form of an integral separately and, therefore, we shall include this case in the form (10.2), bearing in mind that, apart from a finite number of finite terms, the sum can also include an infinite number of infinitesimal terms.

We shall also anticipate that $\kappa_q$ is, in the general case, a complex quantity, as well as $\Omega$ and $F_q$.

Since $F_q(r, \theta)$ is bound to be a periodic function of $\theta$ (for otherwise it would not be single valued), it can be expanded into a Fourier Series with respect to $\theta$,

$$F_q(r, \theta) = \sum_n f_{qn}(r) e^{in\theta}, \quad . \quad . \quad . \quad . \quad (10.3)$$

where $n$ is an integer (positive or negative).

By (10.1)–(10.3),

$$E_x = \sum_q \sum_n E_{xqn}, \qquad B_x = \sum_q \sum_n B_{xqn}, \quad . \quad . \quad (10.4)$$

etc.,

where

$$\left.\begin{aligned} E_{xqn} &= \mathrm{E}_{xqn} e^{i(\kappa_q z + \Omega t + n\theta)}, \\ B_{xqn} &= \mathrm{B}_{xqn} e^{i(\kappa_q z + \Omega t + n\theta)}, \end{aligned}\right\} \quad . \quad . \quad . \quad (10.4a)$$

etc.

$E$ and $B$ must, of course, satisfy the Maxwell equations, and, consequently, as it would not be difficult to show by the introduction of the expressions (10.4) in the Maxwell equations in cylindrical coordinates, $E_{qn}$ and $B_{qn}$

(or, in the case of an infinitesimal $\frac{d}{d\kappa} E_{qn}$ and $\frac{d}{d\kappa} B_{qn}$ for any $\kappa$ in a given interval) must also satisfy the Maxwell equations. But the only solution of the Maxwell equations in the form (10.4*a*), which has a physical meaning, is that which represents a rotating cylindrical wave-train, co-axial with $x$. In order, however, that this wave-train should not undergo any aperiodic transformations in time and that it should retain its shape in the case of uniform movement in the direction $x$, it must be a Whirl (section 4).

Thus, if *a system of e.m. oscillations in free space does not change its structure aperiodically* (*remains stable*) *in the process of uniform translatory motion in a given direction, it must be resolvable into a finite or infinite number of Whirls* (*finite or infinitesimal*) *co-axial with this direction.*

*The phases of all the component Whirls* will undergo, as a result of a translatory motion, a similar transformation in accordance with formula (4.11). The projection of their structure in the $r\theta$ plane will remain invariant, while *all the dimensions* in the $x$ direction, i.e., *in the direction of the translatory motion, will be reduced $\beta$ times*, while *the period of oscillations will increase $\beta$ times*, and *the wave-length will be proportional to $\beta v_x$.*

In view of this proportionality of variation of the dimensions of all the component Whirls, the shape of the e.m. field will not be distorted (apart from the contraction in the direction of motion), but, in accordance with the results obtained in the previous sections, the amplitudes and the energy will be increased $\beta$ times.

When $v_x=0$ ($\beta=1$), the field of each component Whirl is exponentially (and *not* periodically) distributed in the direction of translatory motion. This, however, need not present any real difficulties in resolving the e.m. field into the component Whirls in the case where the distribution of the field in the direction of translatory motion happens to be periodic, because in the same way as the *aperiodic* functions can be presented as the integrals of *periodic* functions (say Fourier Integrals) so the *periodic* ones can be presented as the integrals of *aperiodic* functions.

If we can put $f(x)=\int \varphi(\kappa) e^{i\kappa x} d\kappa$, we can also put $f(ix)=\int \varphi(\kappa) e^{-\kappa x} d\kappa$.

Consider now the case where the Whirl is moving in the axial direction with the uniform speed $v_x$ and corresponding $\beta_x$ and, in a lateral direction, with the speed $v_z$ and corresponding $\beta_z$.

According to the "classical" definition of relative velocity, the system, relatively to which the speed of our Whirl is equal to $v_z$, must itself move with the velocity $v_z$ together with the centre of the Whirl. Therefore, when $v_z=0$, the term of $\phi_1$ depending on time will be the same as it is in formula (4.11), namely, it will be $\frac{\Omega_0 t}{\beta_x}$. When $v_z > 0$ the same term will be $\beta$ times smaller, *i.e.*, it will be $\frac{\Omega_0 t}{\beta_x \beta_z}$ (in accordance with the stability conditions of a system of e.m. oscillations in uniform motion, which were stated above). Thus from the point of view of frequencies

$$\beta=\beta_x\beta_z. \quad . \quad . \quad . \quad . \quad . \quad (10.5)$$

Hence the frequency of the waves forming the Whirl will be

$$\Omega_1=\beta\Omega_0=\beta_x\beta_z\Omega_0. \quad . \quad . \quad . \quad . \quad (10.6)$$

Note that all this reasoning is entirely based on Classical Mechanics and Electrodynamics.

We have just considered the relationship between the axial and lateral motion of the Whirl from the point of view of the *form* of variations of the system, or, as we can say, from the *kinematical* point of view.

Let us now consider the same question from the *dynamical* point of view, *i. e.*, from the point of view of the influence of both motions upon the energy of the Whirl.

As a result of movement of the Whirl in the axial direction, the intensity of its e.m. field will be increased $\beta_x$ times, the energy per unit volume will be increased $\beta_x^2$ times, the volume will be reduced $\beta_x$ times, and, as a result, the total energy will be increased $\beta_x$ times (section 9). In the same way, as a result of motion in the direction $z$, because of the contraction of the field in that direction, the energy will be increased $\beta_z$ times. The total energy,

$$\mathbf{E}=\beta_x\beta_z\mathbf{E}_0=\beta\mathbf{E}_0=\beta\hbar_1\Omega_0=\hbar_1\Omega_1, \quad . \quad . \quad (10.7)$$

whence, again, $\beta=\beta_x\beta_z$.

Hence the dynamical consideration on the classical basis confirms formula (10.5).

The wave-lengths, as well as generally the lengths in each direction, are not interdependent. The "invariant" lengths and the wave-lengths will be

$$\left.\begin{aligned} x_1 &= \beta_x x, & z_1 &= \beta_z z, \\ \lambda_x &= \frac{c^2}{\beta_x \Omega_0 v_x}, & \lambda_z &= \frac{c^2}{\beta_z \Omega_0 v_z}. \end{aligned}\right\} \quad . \quad . \quad (10.8)$$

It may appear reasonable to test our results by the formula for the resultant speed, $v=\sqrt{v_x^2+v_z^2}$. This, however, would not be correct, for we do not know the position of the centres of the component Whirls (co-axial with the direction of the classical resultant of $v_x$ and $v_z$) into which the e.m. field could be resolved. Besides, *for any fixed observer, these axes must rotate, and, consequently, the usual method of deducing the formula* $v=\sqrt{v_x^2+v_z^2}$ *would not be applicable.* Therefore the relationship between the speeds of the Whirl must be determined by their respective contributions to the frequency and the energy (formulæ (10.6) and (10.7)), *i. e.*, by the formula (10.5).

Hence we must define the resultant speed $v$ by means of the formula

$$\beta=\left(1-\frac{v^2}{c^2}\right)^{-1/2}. \quad . \quad . \quad . \quad . \quad (10.9)$$

Consequently, taking into account that

$$\left.\begin{aligned} \beta_x &= \left(1-\frac{v_x^2}{c^2}\right)^{-1/2}, \\ \beta_z &= \left(1-\frac{v_z^2}{c^2}\right)^{-1/2}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (10.9a)$$

we find by (10.5) the following law of addition of the axial and the lateral speeds of the Whirl,

$$v^2=v_x^2+v_z^2-\frac{v_x^2 v_z^2}{c^2}. \quad . \quad . \quad . \quad (10.10)$$

It is different from the classical law of addition of speeds of rigid bodies, and still we obtained it strictly on the basis of classical Electrodynamics, and, therefore, it *is* essentially a classical law of addition of the velocities of the Whirls. As it should be expected in a classical

formula, all the systems of reference associated with this formula have the same units of time and space. The relativistic form of this relationship, in which, in accordance with the Theory of Relativity, the units of space and time are different for each system of reference, will be discussed in section 12.

It would not be difficult to show that formula (10.10) is applicable to any "stable" system of e.m. oscillations (*i. e.*, to a system of e.m. oscillations which does not suffer aperiodic changes in the process of uniform motion). Therefore in this formula the direction $x$ can differ from the direction of the axis of the Whirl.

11. *Electromagnetic Inertia of Whirls.*

The Whirl is essentially a system of e.m. waves in free space, and, therefore, it does not possess any "material" mass in the usually accepted sense, but, nevertheless, it possesses a definite inertia, *i. e.*, it requires energy to acquire speed. According to Classical Dynamics this inertia, whatever may be the mechanism creating it, is measured by the "mass" when this speed is a speed of translatory motion, and by the "moment of inertia" when it is an angular velocity.

The mass $m_{\mathbf{A}}$ in the direction **A** and the moment of inertia $\mathrm{I}_x$ about the axis $x$ of the Whirl, can be expressed in our notations, in accordance with the well-known definitions of Classical Mechanics, in the following way:

$$\left.\begin{aligned} m_{\mathbf{A}} &= \frac{1}{v_{\mathbf{A}}}\frac{d\mathbf{E}}{dv_{\mathbf{A}}} = \frac{1}{v_{\mathbf{A}}}\frac{d(\beta\mathbf{E}_0)}{dv_{\mathbf{A}}}, \\ \mathrm{I}_x &= \frac{1}{\Omega_x}\frac{d\mathbf{E}}{d\Omega_x}. \end{aligned}\right\} \quad . \quad . \quad (11.1)$$

These formulæ also contain the expressions for the momentum $m_{\mathbf{A}}v_{\mathbf{A}}$ and the angular momentum $\mathrm{I}_x\Omega_x$.

Let $\vec{v}$ be resolved into two components $v_x$ and $v_z$ along the axes $x$ and $z$ respectively. Then by (11.1), (10.5), and (10.9),

$$\begin{aligned} m_{\mathbf{A}} &= \frac{\mathbf{E}_0}{v_{\mathbf{A}}}\left(\frac{\partial\beta}{\partial v_x}\frac{dv_x}{dv_{\mathbf{A}}} + \frac{\partial\beta}{\partial v_z}\frac{dv_z}{dv_{\mathbf{A}}}\right) \\ &= \frac{\mathbf{E}_0}{c^2 v_{\mathbf{A}}\, dv_{\mathbf{A}}}\left(\beta_z\beta_x{}^3 v_x\, dv_x + \beta_x\beta_z{}^3 v_z\, dv_z\right) \\ &= \frac{\mathbf{E}_0\beta^3}{c^2 v_{\mathbf{A}}\, dv_{\mathbf{A}}}\left(\frac{v_x\, dv_x}{\beta_z{}^2} + \frac{v_z\, dv_z}{\beta_x{}^2}\right). \quad . \quad . \quad . \quad (11.2) \end{aligned}$$

By (10.10) and (10.9*a*),

$$v\,dv = \frac{v_x\,dv_x}{\beta_x^2} + \frac{v_z\,dv_z}{\beta_z^2}. \quad . \quad . \quad . \quad . \quad . \quad (11.3)$$

Hence (11.2) can be written

$$m_{\mathbf{A}} = \beta^3 \frac{\mathsf{E}_0}{c^2} \frac{v\,dv}{v_{\mathbf{A}}\,dv_{\mathbf{A}}}. \quad . \quad . \quad . \quad . \quad . \quad (11.4)$$

Consider now the case where $v_z=0$ (and, hence, $v=v_x$, $\beta=\beta_x$, $\beta_z=1$).

Then we find from (11.2), that when the direction **A** coincides with the direction $x$ ($v_{\mathbf{A}}=v_x$, $dv_{\mathbf{A}}=dv_x$, $dv_z=0$), *i. e.*, the direction of the acceleration coincides with the direction of the speed,

$$m_{\mathbf{A}}=m_x=\beta^3 \frac{\mathsf{E}_0}{c^2}=\beta^3 m_0. \quad . \quad . \quad . \quad (11.5)$$

When, on the other hand, the direction **A** coincides with $z$ ($v_{\mathbf{A}}=v_z=0$, $dv_{\mathbf{A}}=dv_z$, $dv_z=0$), *i. e.*, the direction of the acceleration is perpendicular to that of the speed, we find from the same formula (11.2)

$$m_{\mathbf{A}}=m_z=\beta \frac{\mathsf{E}_0}{c^2}=\beta m_0. \quad . \quad . \quad . \quad (11.6)$$

We can call $m_x$ the "longitudinal" mass of the Whirl, $m_z$ its "transversal" mass, and

$$m_0=\frac{\mathsf{E}}{c^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.7)$$

which is the mass of the Whirl at rest ($\beta=1$), we can call its "stationary" or "invariant" mass.

Thus in the case of a Whirl the well-known relativistic formula for the mass can be deduced from the principles of Classical Electrodynamics.

As has been already stated *, the Whirl is essentially a rotating system, having the angular velocity

$$\Omega_x=-\frac{1}{n}\Omega_1=-\frac{\beta}{n}\Omega_0 \quad \text{(formulæ (3.6) and (4.8))}$$

and the angular momentum $-2n\hbar_1$. Its moment of inertia

* See the end of section 6.

with respect to the axis $x$ will be, by (11.1), (3.6), and (9.4),

$$I_x = \frac{1}{\Omega_x}\frac{d(\hbar_1\Omega_1)}{d\Omega_x}$$

$$= \frac{n^2\hbar_1}{\Omega_1} = \frac{n^2\hbar_1}{\beta\Omega_0}. \quad . \quad . \quad . \quad . \quad . \quad (11.8)$$

If a system has an angular momentum it must possess gyroscopic properties.

In particular, to rotate the axis $x$ in a plane, with a speed $\Omega_z$, it requires a torque

$$M_z = -2n\hbar_1\Omega_z. \quad . \quad . \quad . \quad . \quad . \quad (11.9)$$

We shall use this formula in Part II.

### 12. *Relativity Relationships in Whirls.*

In order to avoid any ambiguity in the forthcoming discussion, we shall, first of all, put down the terminology which we are going to use.

We call the "The *Principle* of Relativity" the impossibility to detect a motion of a system otherwise than relatively to another system (or "with reference" to another system), or, in other words, the impossibility to detect an absolute motion of a system. If this principle applies to *any* kind of motion, we call it "*General Principle of Relativity*," if it only applies to translatory motion, but does not apply to rotary motion, we call it "*Special Principle of Relativity*."

A theory of physical phenomena which embodies the Principle of Relativity is called "*relativistic*," and if this theory includes all the physical phenomena it is called a "*Theory* of Relativity."

These theories are called "*General*" if they are based on the "General" Principle of Relativity, and "*Special*" if they are based on the Special Principle of Relativity.

We shall see that in any system of Whirls, the General *Principle* of Relativity is a *necessary consequence of their classical electrodynamical features.*

The Special Theory of Relativity, which is appropriate to any system of Whirls, is *the* Einstein Special Theory of Relativity, while a corresponding General Theory of Relativity may be, in principle, different from that developed by Einstein and his School, and must depend

on the actual structure of the system of Whirls to which it is applied.

We shall derive the main relativistic properties of Whirls by using exclusively classical electrodynamical reasoning, and only after this work will have been completed, we shall discuss the connexion of our results with the Theory of Relativity.

Assume that the Whirl moves in the $x$ direction.

Introduce into the expressions of e.m. field of the Whirl the notation

$$\left.\begin{aligned} \xi=\beta q_0|x|, \qquad \rho=q_0 r, \\ t_1=q_0\frac{t}{\beta}-\frac{v_x\xi}{c^2}. \end{aligned}\right\} \quad . \quad . \quad . \quad (12.1)$$

Then the expressions of the phase will become (by (4.11))

$$\left.\begin{aligned} \phi_1=ct_1+n\theta, \\ \phi_2=\chi\xi, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (12.2)$$

while the element of volume will be, in the new units,

$$d\tau=\rho\, d\xi\, d\rho\, d\theta. \quad . \quad . \quad . \quad . \quad . \quad (12.3)$$

The e.m. energy contained in a given "invariant volume" will be

$$\mathsf{E}=A^2\beta q_0\int\frac{d\mathbf{Q}}{d\tau}d\tau. \quad . \quad . \quad . \quad . \quad (12.4)$$

In this formula the space dimensions in the sub-integral expression are expressed in the same units as $\xi$ and $\rho$, but the energy $\mathsf{E}$ itself is expressed in the ordinary units (in which $x$ and $z$ are expressed). However, as one can easily see from this formula, the relative *distribution* of energy in the Whirls is invariant in the new units, and in order that the absolute value should also remain invariant. we have only to include, when we refer to the coordinates $\xi$, $\rho$, and $t_1$, alongside with the space and time units appropriate to these coordinates. a new energy unit which will be $\beta q_0$ times greater than the ordinary unit which we shall use when referring to the coordinates $x$, $r$, and $t$.

Then, in the space coordinates $\xi$, $\rho$, and $\theta$, time co-ordinate $t_1$, and the new energy units, the expressions for the phase and for the distribution of energy density, as

well as the total energy of the Whirl, will become invariant, *not only for any speed of translatory motion, but also for any angular velocity of rotation.*

So far this is only a formal change of appearance of the equations.

This invariancy has, however, a definite physical meaning, as no change of the speed of translatory motion or of the angular velocity of rotation of a Whirl or of a system of Whirls can be detected by physical observation *within* this system. This can only be detected by a *comparative* observation of the system in question and a *fixed* system. The reason is that the standards of space, and time, and energy, which the observer within the moving system may use, will undergo a similar change, for, according to our assumption, our observer lives in the world of Whirls, and all his measuring instruments and standards are also systems of Whirls. For example, he would be unable to discover the motion of the whole system by means of an interferometer, because all the interfering waves, the phases of which could be compared, will undergo a similar change of phase at the point of their interference. Therefore the Michelson's experiment would give in the world of Whirls a negative result.

Let us now consider the influence of the variation of centrifugal force due to the variation of $\Omega_0$ on the general balance of forces. Centrifugal force is proportional to $mr\Omega_0^2$, *i. e.*, proportional to $q_0^2$ (by (12.1), (11.6), (9.4), and (3.11)). The pondermotive electromagnetic force applied to any invariant volume $\int d\tau$ will be proportional to the rate of change of energy of this volume per unit of virtual displacement in the radial direction, in other words, it will be proportional to $\frac{d\mathbf{E}}{dr}$, which in its turn is proportional to $q_0 \frac{d\mathbf{E}}{d\rho}$, *i. e.*, also proportional to $q_0^2$ (by (12.4)).

Hence the centrifugal force due to $\Omega_0$ is proportional to all other forces, and, therefore, an observer living in the system of Whirls, a "moving observer," will be unable to detect any centrifugal force which may be due to the variation of $\Omega_0$.

It may appear that our "moving observer" will notice

the change of direction of magnetic lines relatively to the electric ones, which must depend on $\beta$, as it can be seen from (5.6)–(5.7) (since, by (5.8), $\zeta$ depends on $\beta$). However, he would be unable to discover it, since the lines of force in all the Whirls of the systems which he will have to use for his measurement (for he has no other material at his disposal) will undergo exactly the same change of their direction. Consequently our "moving observer" will express the results of his experience in an "invariant" form, *i. e.*, in terms of $\xi$, $\rho$, and $t_1$, in other words, he will have to put $\beta=1$ and $q_0=1$.

If our "moving observer" were able to "alight" from his moving system and look at all the phenomena from the fixed one, he would find that everything had changed for him, and that he must express his experience in the same way as we did in the previous sections, *i. e.*, using "ordinary" units. He would then naturally decide that while the equations $\beta=1$ and $q_0=1$ are correct from the point of view of the "moving observer," *i. e.*, with reference to a system in which the phenomena occur ("with reference to a moving system"), they are no longer correct from the point of view of a "fixed observer," *i. e.*, an observer relatively to whom the system, in which the phenomena occur, is moving ("with reference to a fixed system"). In the latter case the same phenomena should be interpreted in terms of $x$, $r$, and $t$, and the "invariant" coordinates from (12.1), if used, will have a purely formal meaning, and will be no more really invariant, because the fixed observer will find them depending on $\beta$ and $q_0$.

There is no reason why the fixed observer should be unable to discover the dependence on $\beta$ of the angle between the electric and magnetic vectors, which he was unable to discover when he was moving. He may rightly decide that the direction of these vectors is his only means of orientation, and his geometry may be based on this direction. For instance, if he lives in a gigantic Whirl, in which there are many microscopic ones, which he may use for observation, he may take as a system of coordinates the curvilinear coordinates (a network), which will be formed by intersections of the planes, in which the transversal vector is situated (*i. e.*, the angular planes), with the surfaces on which the longitudinal lines of force are situated. Then his

system of coordinates will continuously rotate. He may, however, not like this, and prefer a four-dimensional coordinate system, in which the time should be included, like a four-dimensional system of coordinates $\phi_1$, $\phi_2$, $\rho$, and $\theta$.

If there is no such dominating Whirl, the observer will probably find it more convenient to base his coordinate system on the average direction and periodic and aperiodic oscillations of the e.m. field.

Anyhow, his system of coordinates will be curvilinear.

We can see that an investigator living in a world of Whirls must build his physical theory on relativistic laws, although the deeper foundation of this basis will still be the Maxwell equations.

Among the consequences of these relativistic properties of Whirls are the law of addition of mutually perpendicular speeds, expressed by the formula (10.10), and the formulæ (11.4)–(11.7) for the mass of the Whirl. This mass $m$ depends on the speed of translational motion (and hence on $\beta$) and on the direction in which the acceleration takes place, in other words, on the direction of the applied force. When, for instance, the direction of acceleration (and hence that of the force) coincides with the direction of the speed, the mass, which is then called the "longitudinal" mass, is proportional to $\beta^3$ (formula (11.5)). On the other hand, when the direction of acceleration (and hence that of the force) is perpendicular to the speed, the mass, which is then called the "transversal" mass, is proportional to $\beta$ (formula (11.6)).

In the case where the force is not perpendicular to the speed there may be some doubt whether it has a real physical meaning, for, unless the relative speed of movement in the direction of the force is zero, it is difficult to say to which system the applied force belongs.

For this reason we shall take the *transversal* mass for a standard and write

$$m=m_z=\beta m_0=\beta\frac{\mathsf{E}_0}{c^2}=\frac{\beta\hbar_1\Omega_0}{c^2}=\frac{\hbar_1\Omega_1}{c^2}=\frac{\hbar_1\beta q_0}{c} \qquad (12.5)$$

It can be seen that, in our study of the properties of Whirls on the basis of Classical Electrodynamics, we have not only come to the relativistic relationships, but in certain parts of our reasoning followed exactly the way

which is familiar in the Einstein Theory of Relativity. We did not point out this parallelism in the course of our reasoning in order to make it absolutely clear that our results were entirely based on Classical Electrodynamics, and the fact that they satisfy the *Principle* of Relativity is *not a postulate, but a corollary.*

Now, after having arrived at our results in a classical electrodynamical way, we can link them with the Einstein Theory of Relativity without fearing confusion.

It is clear that the classical electrodynamic properties of Whirls satisfy not only the Special *Principle* of Relativity, but also the General one, for they imply the impossibility of detecting any kind of absolute motion, rotary as well as translatory motion.

Another result of these properties is that the Whirls satisfy the quantum relationships, which are ascribed by de Broglie to the material waves of elementary particles. But the quantum constant $\hbar_1$, is, generally speaking, only a constant for a given Whirl and not the universal constant. The connexion between $\hbar_1$ and the universal Planck constant $\hbar$ will be discussed later.

The Relativity formulæ for the transformation of coordinates from one system to another are given in (12.1). If in these formulæ we put $q_0=1$, their application will be limited to the case of the relative translatory motion, in other words, they will become Special Relativity formulæ for a system of Whirls. It is the factor $q_0$ which converts these formulæ of the Special Theory of Relativity of Whirls into the formulæ for the transformation of coordinates of the General Theory of Relativity of Whirls.

The Special Relativity formulæ of the Whirl (*i. e.*, formulæ (12.1) with $q_0=1$) are nothing else but the well-known Lorenz–Einstein Transformations of the Einstein Special Theory of Relativity. They were already implied in our result that a physically possible rotating cylindrical wave-train with complex $\Omega$ and $\kappa$ will not suffer aperiodic changes in the process of uniform translatory motion only when the axial dimensions will be reduced $\beta$ times. For this longitudinal contraction (which in the case of Whirls is a corollary of the Maxwell Equations) constituted the fundamental hypothetical assumption from which Lorenz deduced his transformations.

Moreover, the applicability of the Special Theory of Relativity to the Whirls is implied in the very fact that the Whirls are systems of Maxwell e.m. waves, since the Maxwell equations are invariant for the Lorenz-Einstein transformations.

When we analysed the stability conditions of rotating cylindrical waves with complex $\Omega$ and $\kappa$, we also found that, apart from the longitudinal contraction, the de Broglie relationship $c_2 v = c^2$ must be observed. From the de Broglie relationships in conjunction with the Lorenz-Einstein transformations, the quantum conditions can be derived *.

We did not go very deeply into the question of which kind of four dimensional Geometry would be more appropriate for a world of Whirls. We have only stated that it should be, in any case, a "curvilinear" Geometry, which must be somehow connected with the local position of electric and magnetic lines. We have also sketched one of the possibilities for the case of a system in which one "large and powerful" Whirl would be predominant.

In the case of a number of Whirls the Relativistic Geometry would have to average somehow the effect of the structure of the individual Whirls. Its actual form would be a matter of a skilful adaptation to the mutual position and relationship of the Whirls in a system, but not of any hard and fast rules. The question of whether it would be a "stationary" four dimensional world of Einstein, or an evolutionary four dimensional world of de Sitter, would have to be subordinated to the concrete data.

An observer who knew that he lived in a world of Whirls and was familiar with their electromagnetic properties, would consider the World Four Dimensional Geometry, which he would prefer to use in order to simplify the things, only as approximation to express a general electromagnetic effect of a system of Whirls, but he would not take it as a matter of principle which should be either accepted, as an expression of the deepest mysteries of Nature, or fought against. On the other hand, an observer who was not familiar with the electromagnetic properties of Whirls would be probably inclined to regard the four dimensional Relativistic Geometry either as the fundamental principle of his scientific outlook or, alternatively, as a scientific heresy.

* Arthur Haas, 'Wave Mechanics,' London, 1928.

As has been mentioned already in the Introduction, I propose in this paper a hypothesis that the elementary particles like the electron, the proton, the positron, the light quantum, and others, are nothing else but Whirls or their combinations.

The necessary consequence of this hypothesis is that the Principle of Relativity must be valid in the Universe around us, not as a postulate, but as a result of its electromagnetic structure.

## Appendix to Part I.

1. Denote

$$U = \int_{x_1}^{\infty} f(x)\,\varphi(x)\,dx, \quad . \quad . \quad (1.1\text{ap.})$$

$$f^{(-n)}(x) = \underbrace{\int\int \ldots . \int}_{n \text{ times.}} f(x)\,dx^n. \quad . \quad . \quad (1.2\text{ap.})$$

Let U and $f^{(n)}(x)$, at $0 \leq n \leq n_1+1$, be finite. Assume that, when $x \to \infty$, $f^{(n)}(x)$, or $\varphi^{(-n)}(x)$, or both, approach to zero.

Then by repeated integrating in parts,

$$U = \sum_{n=0}^{n_1} (-)^n f^{(n)}(x_1)\,\varphi^{(-n-1)}(x_1) + (-)^{n_1+1} \int_{x_1}^{\infty} f^{(n_1+1)}(x)\varphi^{(-n_1-1)}(x)\,dx. \quad (1.3\text{ap.})$$

Take a particular case where

$$\varphi(x) = e^{(-a \pm bi)x \pm ic}, \quad . \quad . \quad . \quad . \quad (1.4\text{ap.})$$

where $a$ and $b$ are both real and positive.

Then

$$U = \sum_{n=0}^{n_1} \frac{(-)^n f^{(n)}(x_1) e^{(-a \pm ib)x_1 \pm ic}}{(-a \pm ib)^{n+1}} + \frac{(-)^{n_1+1}}{(-a \pm ib)^{n_1+1}} \int_{x_1}^{\infty} f^{(n_1+1)}(x) e^{(-a \pm ib)x \pm ic} dx. \quad (1.5\text{ap.})$$

Take this formula with the + and with the − before $ib$ and $ic$. Add the second modification to and subtract

from the first. We shall obtain, taking into account (1.1 ap.) and (1.4 ap.), after simple transformation,

$$\int_{x_1}^{\infty} f(x)e^{-ax}\begin{matrix}\cos\\ \sin\end{matrix}(bx+c)$$

$$= \frac{1/2}{1/2i} \times \sum_{n=0}^{n_1} \frac{(-)^n f^{(n)}(x_1)e^{-ax_1}[(-a-ib)e^{i(bx_1+c)} \pm (-a+ib)e^{-i(bx_1+c)}]}{(a^2+b^2)^{n+1}}$$

$$+o\left\{(a^2+b^2)^{-\frac{n}{2}-1}\right\}$$

$$= e^{-ax_1}\sum_{n=0}^{n_1}\sum_{m=0}^{n+1}\frac{(-)^{n+m}f^{(n)}(x_1)a^m b^{n-m+1}n!}{m!\,(n-m)!\,(a^2+b^2)^{n+1}}\begin{matrix}\cos\\ \sin\end{matrix}\left[bx_1+c-(m+n-1)\frac{\pi}{2}\right]$$

$$+o\left\{(a^2+b^2)^{-\frac{n}{2}-1}\right\}, \quad \ldots\ldots\ldots \quad (1.6\,\text{ap.})$$

on expansion of $(-a-ib)^{n+1}$ and $(-a+ib)^{n+1}$, according to the binomial formula.

In the case where $f^{(n)}(x)\to 0$ when $x\to\infty$, formula (1.6 ap.) is also applicable when $a=0$ (in accordance with the conditions stated after formula (1.2 ap.)). Then

$$\int_{x_1}^{\infty} f(x)\begin{matrix}\cos\\ \sin\end{matrix}(bx+c)\,dx$$

$$= \sum_{n=0}^{n_1}\frac{(-)^n f^{(n)}(x_1)}{b^n}\cdot\begin{matrix}\cos\\ \sin\end{matrix}\left[bx_1+c-(n-1)\frac{\pi}{2}\right]+o\{b^{-n-1}\}.$$

$$\ldots \quad (1.7\,\text{ap.})$$

2. The $\begin{matrix}\sin\\ \cos\end{matrix}(\sqrt{b_1{}^2x^2+c_1{}^2+b_2x+c_2})$ can be resolved into a series of sin and cos of linear functions of $x$.

This can be done, for example, in the following way :—

First segregate $\begin{matrix}\sin\\ \cos\end{matrix}\sqrt{b_1{}^2x^2+c_1{}^2}$ by means of the formula

$$\begin{matrix}\sin\\ \cos\end{matrix}(m+n)=\begin{matrix}\sin\\ \cos\end{matrix}m\cos n\begin{matrix}+\cos\\ -\sin\end{matrix}m\sin n.$$

Then express $\begin{matrix}\sin\\ \cos\end{matrix}\sqrt{b_1{}^2x^2+c_1{}^2}$ in terms of $J_0(\sqrt{b_1{}^2x^2+c_1{}^2})$ and $J_1(\sqrt{b_1{}^2x^2+c_1{}^2})$ by means of the asymptotic expansion formula (5.10). The $J_0(\sqrt{b_1{}^2x^2+c_1{}^2})$ and $J_1(\sqrt{b_1{}^2x^2+c^2})$ can be expressed in the form of a series of $\begin{matrix}\sin\\ \cos\end{matrix}b_1x$ by using the formulæ

$$\left.\begin{aligned} J_0(\sqrt{b_1^2x^2+c_1^2}) &= \sum_{\kappa=-\infty}^{\infty} (-)^\kappa J_{2\kappa}(c_1)J_{2\kappa}(b_1x), \\ J_1(\sqrt{b_1^2x^2+c_1^2}) &= \frac{\sqrt{b_1^2x^2+c_1^2}}{b_1x} \sum_{\kappa=-\infty}^{\infty} (-)^\kappa J_{2\kappa}(c_1)J_{2\kappa+1}(b_1x). \end{aligned}\right\}$$

. . . (2.1 ap.) *

Using again the asymptotic expansion formulæ (5.10), we can obtain the expression of

$$\begin{matrix}\sin \\ \cos\end{matrix} (\sqrt{b_1^2x^2+c_1^2}+b_2x+c_2)$$

in the form of series of terms containing

$$\begin{matrix}\sin \\ \cos\end{matrix} [b_1x \pm (b_2x+c)].$$

Substituting the result into subintegral expression of

$$\int_{x_1}^{\infty} f(x)e^{-ax} \begin{matrix}\sin \\ \cos\end{matrix} (\sqrt{b_1^2x^2+c_1^2}+b_2x+c_2)\,dx,$$

we can apply for finding this integral the method explained in section (1 ap.).

After integrating we can repeat the process of expressing the $\begin{matrix}\sin \\ \cos\end{matrix}$ functions in terms of Bessel functions used in this section, but in the opposite direction.

In this way we can find the following approximate expressions for the integrals

$$\int_{x_1}^{\infty} f(x)e^{-ax} \begin{matrix}\sin \\ \cos\end{matrix} (\sqrt{b_1^2x^2+c_1^2}+b_2x+c_2)\,dx$$

$$= \begin{cases} \dfrac{f(x_1)e^{-ax_1}}{\sqrt{a^2+(b_1+b_2)^2}} \begin{matrix}\sin \\ \cos\end{matrix} \left(\sqrt{b_1x_1^2+c_1^2}+b_2x_1+c_2-\tan^{-1}\dfrac{b_1+b_2}{a}\right) \\ \quad +o\{[a^2+(b_1+b_2)^2]^{-1}\}, \quad \text{when } a^2+(b_1+b_2)^2 >> 1, \\ e^{-ax_1}\displaystyle\int_{x_1}^{\infty} f(x)\,dx \,.\, \begin{matrix}\sin \\ \cos\end{matrix} (\sqrt{b_1^2x_1^2+c_1^2}+b_2x_1+c_2) \\ \quad \pm o\{[a^2+(b_1+b_2)^2]^{1/2}\}, \quad \text{when } a^2+(b_1+b_2)^2 << 1. \end{cases}$$

. . . (2.2 ap.)

* For the first formula see, for example, Whittaker and Watson, 'A Course of Modern Analysis,' 3rd ed. p. 357, example 4. The second formula can be obtained by differentiating the first, taking into account that

$$J_0(z) = \frac{dJ_1(z)}{dz} \quad \text{and} \quad \frac{dJ_{2\kappa}(z)}{dz} = \tfrac{1}{2}(J_{2\kappa+1}(z)-J_{2\kappa-1}(z)).$$

The process of deducing this formula, which has been here briefly described, is too laborious to be shown here in detail.

It is felt that probably a simpler method of deducing this integration formula can be found.

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XXXI. *On some Properties of the k-Function.* By N. A. Shastri, *M.Sc., Department of Mathematics, College of Science, Nagpur* *.

I.

Professor H. Bateman has defined † the $k$-function as the value of the integral

$$k_n(x)=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos(x\tan\theta-n\theta)\,d\theta,$$

and has obtained its various properties. This paper contains some new properties of the function obtained mostly (except in the case of § 2) by the use of operational methods. In view of the importance of this function in some branches of physics, the following investigations seem to be justifiable.

II. *Summation Formulæ.*

Take one of the recurrence formulæ satisfied by $k_{2n}(x)$ :

$$2xk_{2n}(x)=2nk_{2n}(x)+(n-1)k_{2n-2}(x)+(n+1)k_{2n+2}(x),$$

$$2yk_{2n}(y)=2nk_{2n}(y)+(n-1)k_{2n-2}(y)+(n+1)k_{2n+2}(y).$$

After multiplying them by $k_{2n}(y)$ and $k_{2n}(x)$ respectively, subtract, we get

$$2(x-y)k_{2n}(x)k_{2n}(y)=(n-1)[k_{2n-2}(x)k_{2n}(y)-k_{2n-2}(y)k_{2n}(x)]$$
$$+(n+1)[k_{2n+2}(x)k_{2n}(y)-k_{2n+2}(y)k_{2n}(x)]. \quad . \quad (2.1)$$

Give to $n$ the values 1, 2, 3, . . . successively, multiply the resulting equations by 1, 2, 3, . . . etc., and add, we obtain

$$2(x-y)\sum_{r=1}^{n} rk_{2r}(x)k_{2r}(y)=n(n+1)[k_{2n+2}(x)k_{2n}(y)$$
$$-k_{2n+2}(y)k_{2n}(x)]. \quad . \quad (2.2)$$

* Communicated by the Author.

† Bateman, Trans. Amer. Math. Soc. xxxiii. p. 817.

Substitute $y+z$ for $x$ and expand by Taylor's theorem, we get

$$2z \sum_{r=1}^{n} rk_{2r}(y)\left[\sum_{m=0}^{\infty}\frac{z^m}{m!}k_2^{(m)}(y)\right]$$

$$=n(n+1)\sum_{m=1}^{\infty}\frac{z^m}{m!}[k_{2n}(y)k_{2n+2}^{(m)}(y)-k_{2n+2}(y)k_{2n}^{(m)}(y)].$$

Equate the coefficients of $z^m$ on both sides, we have

$$\sum_{r=1}^{n} rk_{2r}(y)k_{2r}^{(m-1)}(y)$$

$$=\frac{n(n+1)}{2m}[k_{2n}(y)k_{2n+2}^{(m)}(y)-k_{2n+2}(y)k_{2n}^{(m)}(y)]. \quad . \quad (2.3)$$

In particular when $m=1$, we have

$$\sum_{r=1}^{n} r[k_{2r}(y)]^2=\frac{n(n+1)}{2}[k_{2n}(y)k'_{2n+2}(y)-k_{2n+2}(y)k'_{2n}(y)].$$

Hence

$$\sum_{r=1}^{n} r\left[\frac{k_{2r}(y)}{k_{2n}(y)}\right]^2=\frac{n(n+1)}{2}\frac{d}{dy}\left[\frac{k_{2n+2}(y)}{k_{2n}(y)}\right]$$

and

$$\sum_{r=1}^{n} r\left[\frac{k_{2r}(y)}{k_{2n+2}(y)}\right]^2=-\frac{n(n+1)}{2}\frac{d}{dy}\left[\frac{k_{n}(y)}{k_{2n+2}(y)}\right] \quad (2.4)$$

Therefore,

$$\sum_{r=1}^{n} r\int_0^{\infty}\left[\frac{k_{2r}(y)}{k_{2n+2}(y)}\right]^2 dy=-\frac{n(n+1)}{2}\left[\frac{k_{2n}(y)}{k_{2n+2}(y)}\right]_0^{\infty}.$$

When $n$ is an integer

$$k_{2n}(y)=e^{-y}\,[\text{polynomial of degree } n \text{ in } y].$$

Hence

$$\operatorname*{Lt}_{y\to\infty}\frac{k_{2n}(y)}{k_{2n+2}(y)}=0.$$

Again,

$$k_{2n}(0)=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos 2n\theta\, d\theta$$

$$=\frac{\sin n\pi}{n\pi}$$

from Bateman's definition of $k_{2n}(x)$.

Hence

$$\underset{y \to 0}{\text{Lt}} \frac{k_{2n}(y)}{k_{2n+2}(y)} = \frac{n+1}{n} \underset{\theta \to \pi}{\text{Lt}} \frac{\sin n\theta}{\sin (n+1)\theta} \cdot$$
$$= -1.$$

Therefore,

$$\sum_{r=1}^{n} r \int_0^\infty \left[ \frac{k_{2r}(y)}{k_{2n+2}(y)} \right]^2 dy = \frac{n(n+1)}{2}. \quad . \quad . \quad . \quad (2.5)$$

When $m=2$, we get from (2.3)

$$\sum_{r=1}^{n} r k_{2r}(y) k'_{2r}(y) = \frac{n(n+1)}{4} [k_{2n}(y) k''_{2n+2}(y) - k_{2n+2}(y) k''_{2n}(y)]$$
$$= -\frac{n(n+1)}{2} \frac{k_{2n}(y) k_{2n+2}(y)}{y},$$

using the differential equations for $k_{2n+2}(y)$ and $k_{2n}(y)$.

Hence, integrating both sides, we have

$$\sum_{r=1}^{n} r[k_{2r}(y)]^2 = \text{const.} - n(n+1) \int \frac{k_{2n}(y) k_{2n+2}(y)}{y} dy.$$

But $k_{2r}(y)$ is zero both when $y$ is zero and $y$ is infinite if $r$ is an integer. Therefore,

$$\int_0^\infty \frac{k_{2n}(y) k_{2n+2}(y)}{y} dy = 0, \quad . \quad . \quad . \quad (2.6)$$

one of the orthogonal relations satisfied by the $k$-function.

## III.

A given function $h(x)$ is said to be represented operationally by another function $f(p)$, if

$$f(p) = p \int_0^\infty e^{-px} h(x)\, dx, \quad . \quad . \quad . \quad . \quad (3.1)$$

provided that the integral converges. The relation between the " original " $h(x)$ and its " image " $f(p)$ is denoted by

$$f(p) \doteqdot h(x).$$

The following are a few of the rules and the results proved by Carson and van der Pol * :

* Balth. van der Pol and Niessen, Phil. Mag. xiii. p. 539 (1932).

$$f\left(\frac{p}{s}\right) \doteqdot h(sx), \quad \text{if } s=\text{const.}>0, \quad . \quad . \quad \text{(R 1)}$$

$$\frac{p}{p+\alpha} f(p+\alpha) \doteqdot e^{-\alpha x} h(x). \quad . \quad . \quad . \quad \text{(R 2)}$$

The product theorem

$$\frac{1}{p} f_1(p) . f_2(p) \doteqdot \int_0^x h_1(\xi) h_2(x-\xi)\, d\xi$$

$$\doteqdot \int_0^x h_2(\xi) h_1(x-\xi)\, d\xi, \quad . \quad \text{(R 3)}$$

where $h_1(x)$ and $h_2(x)$ are the originals of $f_1(p)$ and $f_2(p)$ respectively.

$$\frac{1}{p^n} \doteqdot \frac{x^n}{\Pi(n)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(F 1)}$$

$$\frac{p}{p^2+1} \doteqdot \sin x, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(F 2)}$$

$$\frac{p^2}{p^2+1} \doteqdot \cos x, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(F 3)}$$

$$p^{-n} e^{-\frac{1}{p}} \doteqdot x^{\frac{n}{2}} \mathrm{J}_n(2\sqrt{x}). \quad . \quad . \quad . \quad . \quad \text{(F 4)}$$

## IV.

The differential equation satisfied by $k_{2n}(x)$ is

$$x \frac{d^2y}{dx^2} + (2n-x)y = 0.$$

Let

$$\mathrm{X} = \int_0^\infty e^{-px} y(x)\, dx.$$

Multiplying the differential equation by $e^{-px}$ and integrating and noticing that $k_{2n}(0)=0$ for integral values of $n$, we obtain the differential equation satisfied by X, viz.,

$$(1-p^2)\frac{d\mathrm{X}}{dp} = (2p-2n)\mathrm{X},$$

of which the solution is

$$\mathrm{X} = \frac{\mathrm{C}(1-p)^{n-1}}{(1+p)^{n+1}}.$$

Adjusting C properly, we find

$$\frac{2p(1-p)^{n-1}}{(1+p)^{n+1}} \doteqdot k_{2n}(x). \quad . \quad . \quad . \quad . \quad (4.1)$$

Similarly,

$$\frac{p}{1+p} \doteqdot k_0(x). \quad . \quad . \quad . \quad . \quad . \quad (4.2)$$

By a similar process *, we get

$$\frac{(-)^n\sqrt{\pi}\Gamma(2n+2)}{2^n\,\Gamma(n+1)} \frac{p(p-1)^n}{(p+1)^{n+\frac{3}{2}}} \doteqdot \mathrm{D}_{2n+1}(2\sqrt{x}), \quad (4.3)$$

and

$$\frac{2\pi}{\Gamma(n+1)} \frac{\sqrt{p}(p-1)^n}{(p+1)^{n+1}} \doteqdot i[\mathrm{D}^2_{-(n+1)}(i\sqrt{2x}) - \mathrm{D}^2_{-(n+1)}(-i\sqrt{2x})].$$

$$. \quad . \quad . \quad (4.4)$$

## V. *Expansions.*

Combining (R 1) with (F 3) we have

$$\frac{p^2}{p^2+\sin^2\theta} \doteqdot \cos(x\sin\theta).$$

Using (R 2),

$$\frac{p(p+\cos\theta)}{p^2+2p\cos\theta+1} \doteqdot e^{-x\cos\theta}\cos(x\sin\theta).$$

Now from (4.1) and (4.2),

$$\frac{p}{p+1} + \sum_{r=1}^{\infty} \frac{(-)^r 2p(1-p)^{2r-1}}{(1+p)^{2r+1}} \left[\tan\frac{\theta}{2}\right]^{2r}$$

$$\doteqdot \sum_{r=0}^{\infty} (-)^r k_{4r}(x) \left[\tan\frac{\theta}{2}\right]^{2r}.$$

The left-hand side is equal to

$$\frac{p}{1+p}\cdot\left[\frac{p^2+2p\cos^2\frac{\theta}{2}+\cos\theta}{1+p^2+2p\cos\theta}\right] = \frac{p(p+\cos\theta)}{1+2p\cos\theta+p^2},$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

* Mitra, Proc. Edin. Math. Soc. (2) iv. (1934).

Hence

$$e^{-x\cos\theta}\cos[x\sin\theta] = \sum_{r=0}^{\infty}(-)^r k_{4r}(x)\left[\tan\frac{\theta}{2}\right]^{2r}. \quad (5.1)$$

Substituting $a=\tan\frac{\theta}{2}$, we get

$$\left.\begin{aligned} e^{x\frac{a^2-1}{a^2+1}}\cos\left[\frac{2ax}{1+a^2}\right] &= \sum_{r=0}^{\infty}(-)^r k_{4r}(x)a^{2r}, \qquad -1<a<1. \\ \text{By a similar method we obtain} \\ e^{x\frac{a^2-1}{a^2+1}}\sin\left[\frac{2ax}{1+a^2}\right] &= \sum_{r=0}^{\infty}(-)^r k_{4r+2}(x)a^{2r+1}, \quad -1<a<1. \end{aligned}\right\} \quad . \quad . \quad . \quad (5.2)$$

Again,

$$\frac{p^2}{p^2+\tan^2\theta} \doteqdot \cos(x\tan\theta); \qquad \frac{p\tan\theta}{p^2+\tan^2\theta} \doteqdot \sin(x\tan\theta). \quad (5.3)$$

Now

$$\frac{p}{p+1}+\sum_{1}^{\infty}\frac{2p(1-p)^{n-1}}{(1+p)^{n+1}}e^{i2n\theta} = \frac{p^2}{p^2+\tan^2\theta}+i\frac{p\tan\theta}{p^2+\tan^2\theta}.$$

Term by term interpretation of both sides with the help of (4.1) and (5.3), gives us

$$\sum_{0}^{\infty}k_{2n}(x)e^{2in\theta} = \cos(x\tan\theta)+i\sin(x\tan\theta).$$

Therefore,

$$\left.\begin{aligned} &\sum_{0}^{\infty}k_{2n}(x)\cos 2n\theta = \cos(x\tan\theta) \\ \text{and} \quad &\sum_{0}^{\infty}k_{2n}(x)\sin 2n\theta = \sin(x\tan\theta). \end{aligned}\right\} \quad . \quad . \quad (5.4)$$

In particular, when $\theta=\frac{\pi}{4}$, we have

$$\left.\begin{aligned} &\sum_{0}^{\infty}(-)^r k_{4r}(x) = \cos x \\ \text{and} \quad &\sum_{0}^{\infty}(-)^r k_{4r+2}(x) = \sin x, \\ \text{and when } \theta=0, \\ &\sum_{r=0}^{\infty}k_{2n}(x) = 1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5.5)$$

These results (5.5) have been obtained by Bateman by a different method.

Combining (R 2) with (4.1) we have

$$\frac{(-)^{n-1}2p^n}{(p+2)^{n+1}} \doteq e^{-x}k_{2n}(x). \quad . \quad . \quad . \quad . \quad . \quad (5.6)$$

Apply (R 1) to (5.6) we obtain

$$\frac{(-)^{n-1}2p^n}{(p\lambda+2)^{n+1}} \doteq \lambda^{-n}e^{-\frac{x}{\lambda}}k_{2n}\left(\frac{x}{\lambda}\right), \quad \lambda>0 \text{ and constant.}$$

Again,

$$\frac{(-)^{n-1}2p^n}{(p\lambda+2)^{n+1}} = \frac{(-)^{n-1}2p^n}{(p+2)^{n+1}}\left[1+\frac{p(\lambda-1)}{p+2}\right]^{-n-1}$$

$$= \sum_{r=0}^{\infty} \frac{\Gamma(n+r+1)}{\Gamma(n+1)\,\Gamma(r+1)} \cdot \frac{(\lambda-1)^r(-)^{n+r-1}2p^{n+r}}{(p+2)^{n+r+1}},$$

$$0<\lambda<2.$$

From the term by term interpretation of both sides we get

$$\lambda^{-n}e^{-\frac{x}{\lambda}}k_{2n}\left(\frac{x}{\lambda}\right) = e^{-x}\sum_{r=0}^{\infty} \frac{\Gamma(n+r+1)}{\Gamma(n+1)\Gamma(r+1)} \cdot (\lambda-1)^r k_{2n+2r}(x),$$

$$. \quad . \quad . \quad (5.7)$$

where $0<\lambda<2$ and $n$ is a positive integer.

## VI. *Relations between the k-function and Parabolic Cylinder Functions.*

Now

$$\frac{2p(p-1)^{n-1}}{(p+1)^{n+1}} = \frac{\sqrt{2}.p(p-1)^{n-1}}{(p+1)^{n+\frac{1}{2}}}\left[1-\frac{p-1}{p+1}\right]^{\frac{1}{2}}$$

$$= \sqrt{2}.\left[\frac{p(p-1)^{n-1}}{(p+1)^{n+\frac{1}{2}}} - \sum_{r=1}^{\infty} \frac{1.3.5\ldots(2r-3)}{2^r\Gamma(1+r)} \frac{p(p-1)^{n+r-1}}{(p+1)^{n+r+\frac{1}{2}}}\right].$$

Interpreting both sides with the help of (4.1) and (4.3) we have

$$k_{2n}(x) = \frac{2^{n-\frac{1}{2}}}{\sqrt{\pi}}\left[\frac{\Gamma(n)}{\Gamma(2n)} \cdot \mathrm{D}_{2n-1}(2\sqrt{x})\right.$$

$$\left. + \sum_{r=1}^{\infty}(-)^{r+1}\frac{1.3.5\ldots(2r-3).\Gamma(n+r)}{\Gamma(r+1)\,\Gamma(2n+2r)}\mathrm{D}_{2n+2r-1}(2\sqrt{x})\right].$$

$$. \quad . \quad . \quad (6.1)$$

Conversely,

$$\frac{p(p-1)^n}{(p+1)^{n+\frac{3}{2}}} = \frac{\sqrt{2}p(p-1)^n}{(p+1)^{n+2}} \cdot \left[1-\frac{p-1}{p+1}\right]^{-\frac{1}{2}}$$

$$= \sqrt{2} \cdot \left[\frac{p(p-1)^n}{(p+1)^{n+2}} + \sum_{r=1}^{\infty} \frac{1.3.5 \ldots (2r-1)}{2^r \Gamma(1+r)} \frac{p(p-1)^{n+r}}{(p+1)^{n+r+2}}\right].$$

Interpreting as above, we get

$$D_{2n+1}(2\sqrt{x}) = \sqrt{2\pi} \cdot \frac{\Gamma(2n+2)}{\Gamma(n+1)} \left[\frac{k_{2n+2}(x)}{2^{n+1}} + \sum_{r=1}^{\infty} \frac{(-)^r 1.3.5 \ldots (2r-1)}{\Gamma(1+r) \cdot 2^{n+r+1}} k_{2n+2r+2}(x)\right]. \quad (6.2)$$

Again,

$$\frac{\sqrt{p}(p-1)^n}{(p+1)^{n+1}} = \frac{2p(p-1)^n}{(p+1)^{n+2}} \left[1-\left(\frac{p-1}{p+1}\right)^2\right]^{-\frac{1}{2}}$$

$$= \left[\frac{2p(p-1)^n}{(p+1)^{n+2}} + \sum_{r=1}^{\infty} \frac{1.3.5 \ldots (2r-1)}{2^r \Gamma(1+r)} \cdot \frac{2p(p-1)^{n+2r}}{(p+1)^{n+2r+2}}\right].$$

Interpret both sides with the help of (4.4) and (4.1) respectively, we obtain

$$\frac{(-)^n \Gamma(1+n) i}{2\pi} [D^2_{-(n+1)}(i\sqrt{2x}) - D^2_{-(n+1)}(-i\sqrt{2x})]$$

$$= \left[k_{2n+2}(x) + \sum_{r=1}^{\infty} \frac{1.3.5 \ldots (2r-1)}{2^r \Gamma(1+r)} k_{2n+4r+2}(x)\right]. \quad (6.3)$$

Conversely,

$$\frac{2p(p-1)^{n-1}}{(p+1)^{n+1}} = \frac{\sqrt{p}(p-1)^{n-1}}{(p+1)^n} \left[1-\left(\frac{p-1}{p+1}\right)^2\right]^{\frac{1}{2}}$$

$$= \left[\frac{\sqrt{p}(p-1)^{n-1}}{(p+1)^n} - \sum_{r=1}^{\infty} \frac{1.3.5 \ldots (2r-3)}{2^r \Gamma(1+r)} \cdot \frac{\sqrt{p}(p-1)^{2r+n-1}}{(p+1)^{2r+n}}\right].$$

Interpreting as above, we get

$$(-)^{n-1}k_{2n}(x) = \left[\frac{\Gamma(n)i}{2\pi}\{\mathrm{D}^2_{-n}(i\sqrt{2x}) - \mathrm{D}^2_{-n}(-i\sqrt{2x})\}\right.$$
$$- \sum_{r=1}^{\infty} \frac{1.3.5\ldots(2r-3)}{2^r\Gamma(1+r)} \cdot \frac{\Gamma(n+2r)i}{2\pi}$$
$$\left.\times[\mathrm{D}^2_{-(2r+n)}(i\sqrt{2x}) - \mathrm{D}^2_{-(2r+n)}(-i\sqrt{2x})]\right]. \quad \ldots \quad (6.4)$$

Finally,

$$\frac{1}{p}\cdot\frac{p(p-1)^{n-1}}{(p+1)^{n+1}}\cdot\frac{p(p-1)^m}{(p+1)^{m+\frac{3}{2}}} = \frac{p(p-1)^{m+n-1}}{2^{\frac{3}{2}}(p+1)^{m+n+1}}\left[1-\frac{p-1}{p+1}\right]^{\frac{3}{2}}$$
$$= \frac{1}{2^{\frac{3}{2}}}\left[\frac{p(p-1)^{m+n-1}}{(p+1)^{m+n+1}}\right.$$
$$\left. + \sum_{r=1}^{\infty} \frac{3.1.1.3.5\ldots(2r-5)}{2^r\Gamma(1+r)} \frac{p(p-1)^{m+n+r-1}}{(p+1)^{m+n+r+1}}\right].$$

Using the product theorem and interpreting with the help of (4.1) and (4.3), we obtain

$$\int_0^x k_{2n}(\xi)\mathrm{D}_{2m+1}(2\sqrt{x-\xi})\,d\xi$$
$$= \frac{\sqrt{\pi}\Gamma(2m+2)}{\Gamma(m+1)}\frac{1}{2^{m+\frac{3}{2}}}\left[k_{2m+2n}(x)\right.$$
$$\left. + \sum_{r=1}^{\infty} \frac{3.1.1.3.5\ldots(2r-5)}{2^r\Gamma(1+r)}(-)^r k_{2m+2n+2r}(x)\right]. \quad (6.5)$$

## VII. *Miscellaneous Results.*

Combining (4.1) with (R 2) and (F 4) with (R 1), we get

$$\frac{2p^n(-)^{n-1}}{(p+2)^{n+1}} \doteqdot e^{-x}k_{2n}(x)$$

and
$$\frac{2s}{p}e^{-\frac{2s}{p}} \doteqdot \sqrt{2xs}\,\mathrm{J}_1(2\sqrt{2xs}).$$

Now

$$\frac{2p^n(-)^{n-1}}{(p+2)^{n+1}} = \frac{2(-)^{n-1}}{p}\cdot\left(\frac{p}{p+2}\right)^{n+1}$$
$$= \frac{(-)^{n-1}}{\Gamma(n+1)}\int_0^{\infty} e^{-s}s^{n-1}\cdot\frac{2s}{p}e^{-\frac{2s}{p}}\,ds.$$

Interpret both sides with the help of the relations obtained above, we get

$$k_{2n}(x) = \frac{(-)^{n-1}}{\Gamma(1+n)} \int_0^\infty e^{x-s} s^{n-1} \sqrt{2xs} \,.\, \mathrm{J}_1(2\sqrt{2xs})\, ds$$

$$\text{or} \qquad = \frac{2(-)^{n-1}}{\Gamma(1+n)} \int_0^\infty e^{x-\frac{t^2}{2x}} \left(\frac{t^2}{2x}\right)^n \mathrm{J}_1(2t)\, dt. \quad . \quad . \quad (7.1)$$

Put $\frac{1}{2x} = z$, we get

$$\frac{(-)^{n-1}}{z^n} \,.\, e^{-\frac{1}{2z}} k_{2n}\left(\frac{1}{2z}\right) \Gamma(1+n)$$

$$= 2\int_0^\infty e^{-zt^2} t^{2n} \mathrm{J}_1(2t)\, dt \quad . \quad . \quad . \quad (7.2)$$

$$= \frac{\Gamma(1+n)}{z^{n+1}} \,.\, {}_1\mathrm{F}_1\left(n+1,\, 2,\, -\frac{1}{z}\right) *$$

$$= \sum_{m=0}^{\infty} \frac{(-)^m \Gamma(n+m+1)}{\Gamma(m+1)\,\Gamma(m+2)} \frac{1}{z^{m+n+1}} .$$

Therefore,

$$(-)^{n-1} e^{-z} k_{2n}(x) = \sum_{m=0}^{\infty} \frac{(-)^m \Gamma(n+m+1)}{\Gamma(n+1)\,\Gamma(m+1)\,\Gamma(m+2)} \,.\, (2x)^{m+1},$$

$$\left[\frac{1}{2z} = x\right],$$

$$= 2x \,.\, {}_1\mathrm{F}_1[n+1,\, 2,\, -2x]. \quad . \quad . \quad . \quad . \quad (7.3)$$

Substitute $t^2 = 2x$ in (7.2) and simplify, and replace $z$ by $\frac{p}{2}$, we get

$$(-)^{n-1} 2^{-\frac{1}{2}} \Gamma(1+n) \frac{1}{p^{n-\frac{1}{2}}} \,.\, e^{-\frac{1}{p}} k_{2n}\left(\frac{1}{p}\right)$$

$$= p\int_0^\infty e^{-px} x^{n-\frac{1}{2}} \mathrm{J}_1(2\sqrt{2x})\, dx.$$

Therefore, from (3.1), we have

$$(-)^{n-1} 2^{-\frac{1}{2}} \Gamma(1+n) \frac{1}{p^{n-\frac{1}{2}}} e^{-\frac{1}{p}} k_{2n}\left(\frac{1}{p}\right) \doteq x^{n-\frac{1}{2}} \mathrm{J}_1(2\sqrt{2x}). \quad (7.4)$$

* Watson, 'Bessel Functions,' 13.3 (2).

Now

$$\int_0^\infty e^{-\alpha\frac{s^2}{4}} s^{n+1} J_n(s)\, ds = \frac{2^{n+1}}{\alpha^{n+1}} e^{-\frac{1}{\alpha}}$$

$$= 2^{n+1} \sum_{r=0}^{\infty} \frac{(-)_r}{\Gamma(1+r)} \frac{1}{\alpha^{n+r+1}}.$$

Put $\alpha = \frac{p+2}{p}$, we get

$$\int_0^\infty e^{-\frac{s^2}{4}} s^{n-1} J_n(s) . e^{-\frac{s^2}{2p}} . \frac{s^2}{2p} ds = 2^n \sum_{r=0}^{\infty} \frac{(-)^r p^{n+r}}{\Gamma(1+r)(p+2)^{n+r+1}}.$$

Again,

$$\frac{s^2}{2p} e^{-\frac{s^2}{2p}} \doteq \left(\frac{s^2 x}{2}\right)^{\frac{1}{2}} J_1(\sqrt{2xs^2}).$$

Interpreting both sides by using this relation and (5.6), we obtain

$$\int_0^\infty e^{-\frac{s^2}{4}} s^n (2x)^{\frac{1}{2}} J_n(s) J_1(\sqrt{2x}.s)\, ds$$

$$= 2^n \sum_{r=0}^{\infty} \frac{(-)^{n-1} e^{-x}}{\Gamma(1+r)} k_{2n+2r}(x). \quad (7.5)$$

Finally,

$$\frac{1}{p} \cdot \frac{2p(1-p)^{n-1}}{(1+p)^{n+1}} \cdot \frac{2p(1-p)^{m-1}}{(1+p)^{m+1}}$$

$$= \frac{p(1-p)^{m+n-2}}{(1+p)^{m+n}} \cdot \left[1 + \frac{1-p}{1+p}\right]^2$$

$$= \frac{p(1-p)^{m+n-2}}{(1+p)^{m+n}} + \frac{2p(1-p)^{m+n-1}}{(1+p)^{m+n+1}} + \frac{p(1-p)^{m+n}}{(1+p)^{m+n+2}}.$$

Using the product theorem (R 3) and interpreting, we have

$$\int_0^x k_{2n}(\xi) k_{2m}(x-\xi)\, d\xi$$

$$= \int_0^x k_{2m}(\xi) k_{2n}(x-\xi)\, d\xi$$

$$= \tfrac{1}{2} . [k_{2m+2n-2}(x) + 2k_{2m+2n}(x) + k_{2m+2n+2}(x)]. \quad (7.6)$$

XXXII. *The Isotopes of Arsenic, Selenium, Mercury, and Lead.* *By* H. J. Walke, *M.Sc., A.Inst.P., Demonstrator in Physics, Washington Singer Laboratories, University College, Exeter* *.

IT has been suggested in recent papers † that the elements have been evolved by a process of neutron addition, short-lived $\beta$-radioactive synthesis links (which are the "missing" isotopes) bridging the various elements. Assuming this theory of neutron capture it appears that the isotopic constitution of mercury and lead need further investigation.

The synthesis of thallium from mercury should result from the $\beta$-radioactivity of $Hg_{80}^{203}$ and $Hg_{80}^{205}$, both of which are assumed to be unstable (and should therefore be missing), so that we have the reactions

$$Hg_{80}^{203} \rightarrow \beta \rightarrow Tl_{81}^{203}; \; Hg_{80}^{205} \rightarrow \beta \rightarrow Tl_{81}^{205}.$$

Aston, however, has found that $Hg_{80}^{203}$ exists in very small abundance. Synthesis from thallium to lead should, therefore, be due to the $\beta$-radioactivity of the missing $Tl_{81}^{204}$ and $Tl_{81}^{206}$, and in the direct synthesis $Pb_{82}^{204}$ should, therefore, be the lightest lead isotope. But since $Pb_{82}^{203}$ apparently exists in small abundance, $Tl_{81}^{203}$ may be spontaneously $\beta$-radioactive, viz. :—

$$Hg_{80}^{203} \rightarrow \beta \rightarrow Tl_{81}^{203} \rightarrow \beta \rightarrow Pb_{82}^{203}.$$

It has been shown, however ‡, that the presence of certain isotopes of lower mass numbers than would be expected in the direct synthesis of the isotopes by neutron capture, can be explained by means of a secondary process involving the emission of excess energy in the form of kinetic energy of an $\alpha$-particle from a few nuclei newly formed as a result of a $\beta$-ray transformation, *e.g.*,

$$Br_{35}^{82} \rightarrow \beta \rightarrow \overset{*}{K}r_{36}^{82} \rightarrow \alpha \rightarrow \overset{*}{S}e_{34}^{78} \rightarrow \beta \rightarrow Br_{35}^{78} \rightarrow \beta \rightarrow Kr_{36}^{78}.$$

* Communicated by Prof. F. H. Newman, D.Sc.

† Walke, Phil. Mag. xviii. p. 795 (1934); xix. p. 33 (1935); xix. p. 341 (1935).

‡ Walke, Phil. Mag. xix. p. 33 (1935).

This process resembles the abnormal disintegration of some of the spontaneously radioactive elements, in which the emission of a long range α-particle occurs from a small number of nuclei formed in states of excess energy usually as a result of an initial β-ray transformation, *i. e.*, $RaC^1$. It involves the emission of α-particles from a small percentage of normally stable nuclei, when these are formed in a state of excess energy. Thus it is possible that $Tl_{81}^{203}$ is stable and that an additional, undetected, stable lead isotope of mass number 202 exists. This could be formed during the synthesis of $Pb_{82}^{206}$ :

$$Tl_{81}^{206} \rightarrow \beta \rightarrow \overset{*}{P}b_{82}^{206} \rightarrow \gamma \rightarrow Pb_{82}^{206}.$$

Alternatively in a small number of such nuclei :

$$\overset{*}{P}b_{82}^{206} \rightarrow \alpha \rightarrow \overset{*}{H}g_{80}^{202} \rightarrow \beta \rightarrow Tl_{81}^{202} \rightarrow \beta \rightarrow Pb_{82}^{202}.$$

Hence, if $Pb_{82}^{202}$ exists as a stable isotope, $Pb_{82}^{203}$ could be formed (as is $Pb_{82}^{205}$ from $Pb_{82}^{204}$), by neutron addition.

If, therefore, all the isotopes of mercury and lead have been detected, $Hg_{80}^{203}$ and $Tl_{81}^{203}$ should be weakly β-radioactive. If, however, $Pb_{82}^{202}$ exists and $Hg_{80}^{203}$ is missing, mercury and thallium are stable elements.

In a similar manner the presence of $Se_{34}^{74}$ was explained * by assuming that arsenic had a lighter isotope $As_{33}^{73}$ (as suggested by the new atomic weight 74·91), $Se_{34}^{74}$ being formed as a result of the β-radioactivity of the missing $As_{33}^{74}$. This hypothesis involved the spontaneous β-radioactivity of $Ge_{32}^{73}$. An alternative explanation is that $As_{33}^{77}$ (the absence of which from the mass spectrum is due to it being β-radioactive and having disappeared to form $Se_{34}^{77}$) is of sufficiently long life to allow the formation of $As_{33}^{78}$ by neutron capture during the stellar synthesis of the elements. In this case some of the $\overset{*}{S}e_{34}^{78}$ nuclei newly formed in a state of excess energy as a result of the β-radioactivity of the missing $As_{33}^{78}$ emit an α-particle, and

* Walke, *loc. cit.*

$Se_{34}^{74}$ is produced as a result of the secondary nuclear disintegration synthesis process postulated, viz. :

$$As_{34}^{78} \rightarrow \beta \rightarrow \overset{*}{Se}_{34}^{78} \rightarrow \alpha \rightarrow \overset{*}{Ge}_{32}^{74} \rightarrow \beta \rightarrow As_{33}^{74} \rightarrow \beta \rightarrow Se_{34}^{74}.$$

Thus the presence of $Se_{34}^{74}$ can be explained by the present known isotopic constitution of germanium and arsenic. According to this theory of the formation of $Se_{34}^{74}$, $Ge_{32}^{72}$ is stable and arsenic is a single element of mass number 75. This implies that the chemical value of the atomic weight of arsenic is too low.

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## XXXIII. *The Osmotic Pressures of a Mixed Vapour.* *By* The Earl of Berkeley, *F.R.S.**

### *Introduction.*

THE dominant feature in osmotic pressure work is the dearth of satisfactory semipermeable membranes. This makes it certain that for the vast majority of solutions and mixed vapours indirect methods must be sought.

For solutions containing one volatile constituent the approximate formula, given by Berkeley and Hartley, and later made by Prof. Porter an exact equation, has been sufficient. For research on solutions of two volatile constituents another equation given by Prof. Porter †, in which the significant factor is the experimental determination of the partial pressure of the solvent in the mixed vapour, is confined to vapours that follow the gas laws. This equation is modified by Berkeley and Burton ‡ so as to be exact, and thus to include all possible cases. The equation is independent of the partial pressures ; but to use it, a knowledge of the pressure of the pure vapour, which is in osmotic equilibrium with the mixed vapour, is essential. Since the lack of membrane

* Communicated by the Author.

† "On the Osmotic Pressure of Compressible Solutions," Part II. Proc. Roy. Soc. 1908, p. 457.

‡ "Contribution to the Osmotic Theory of Solutions," Phil. Mag. 1909, p. 605.

rules out a direct experiment, the quantity must be found by calculation from other data.

The main object of this communication is to show how this can be done. The equations to be used do not involve the components' specific volumes *in* the solution (or mixed vapour). This may be a distinct gain, for, among other things, it is no longer necessary to determine the compressibilities of the solutions.

Prof. Porter's formula, as given in his paper, applies to a mixed vapour that conforms to a certain condition (the condition is shown later on); but it depends on experiments, which are both difficult and lacking in precision, for finding the partial pressures. It will be seen that, if the "gas" law obeyed by the mixed vapour is determined, the partial pressures are not required; and, further, as this "gas" law is essential to the application of Berkeley and Burton's equation a saving of labour is attained.

Attention should be drawn to a point which arises when considering solutions of two volatile components. Zawidsky * has shown that, in many cases, the concentration of the solution is not the same as that of its vapour; and the question arises whether this phenomenon is a function of the pressure put upon the solution. In other words, does the change in this pressure, when the solution is separated from its vapour by a vapour permeable membrane, alter the concentration in both phases?

At present there does not seem to be any experiment that answers the question †; so partly on account of this possibility, and partly so as to avoid condensation taking place when changing the pressure, a somewhat unusual thermodynamic cycle is detailed.

In the course of the discussion it will be realized that some of the methods advocated for solving the problems are not exact, and therefore must end in approximations; in certain circumstances they apparently lead to somewhat indefinite values. These indefinite values may, in most cases, be settled if data for a large enough series of solutions and mixed vapours have been obtained.

Methods for finding osmotic pressures of solutions other than by Berkeley and Burton's modification of Porter's equation are given because in some cases they may be more convenient; also by means of one of them

* *Zeit. Phys. Chem.* vol. xxxv. pp. 129–203.

† See Appendix.

the changes in the pressure of the mixed vapour caused by changes in the pressure on the solution can be determined provided no changes in the concentrations take place.

*The Notation.* The symbols used are :—

$\pi_1$ and $\phi_1$ for the partial pressures of solvent and solute (respectively) in the mixed vapour.

$\pi_0$ and $\phi_0$ for the vapour pressures of solvent and solute respectively.

$C_z$ and $C_v$ for the concentrations in the solution and mixed vapour respectively ; and are the number of grammes of solvent to one gramme of solute.

$C_1$ and $C_2$. In some cases it is more convenient to use these symbols to represent the concentrations ; they are the number of grammes of solvent and solute (respectively) per gramme of mixture ; the context will show whether they apply to liquids or vapours.

$w_z$ and $w_v$ for the specific volumes of the solution and mixed vapour (respectively) stratified by by gravity and diffusion.

$S_z$ and $S_v$ for the specific volumes of the solution and mixed vapour (respectively) stratified by gravity only *.

$s_z$ and $s_v$ for the specific volumes of the solvent in the pure liquid and vapour phases respectively.

$\sigma_z$ and $\sigma_v$ for the specific volumes of the solute in the liquid and vapour phases respectively.

$s_1$ and $\sigma_1$ are defined thus : $s_1$ is the decrease in the volume of a large mass of mixed vapour (or solution) when one gramme of solvent is withdrawn, while $\sigma_1$ is the corresponding quantity for the solute. The two quantities will be referred to as the specific volume of the solvent or solute *in* the mixed vapour or solution (the context will show to which phase they apply).

* The meaning attached to the phrase "stratified by gravity only" is explained on p. 492.

The following is a summary of the factors required for the various discussions :—

It is assumed that $\pi_1+\phi_1$ has been determined experimentally for a series of solutions of known concentrations ; that $\pi_0$ and $\phi_0$ have also been found ; that the specific volumes and "compressibilities" of all three liquids and their vapours are known ; and, further, that during all operations the temperature remains constant. It is also assumed that when any of the pure liquids are subjected to large vertical displacements their vapour pressures in the new positions are known.

*Condition dividing Mixed Vapours into two Classes.*

All mixed vapours must come under one or other of the following heads :—

(1) The pure vapours obey the gas law ; and when mixed, (*a*) the components still follow the gas law or, (*b*), one or both components obey a law other than that of perfect gases.

(2) The pure vapours obey other laws than the gas law ; and when mixed, (*a*) they still follow the same laws as before or, (*b*), one or both components conform to different laws.

(3) A chemical reaction occurs when the vapours are mixed. This type is not considered further *, for obviously a third component must be present.

*Case* (1) (*a*).—Here it is clear that when a pure vapour is in osmotic equilibrium with the mixed vapour, its partial pressure in the mixture is the same as its pressure outside. (Dalton's law.)

*Cases* (1) (*b*) and (2) (*b*).—Here it is evident that the partial pressure is not the same as the pressure of the pure vapour.

*Case* (2) (*a*).—It was thought at first that this case was similar to (1) (*a*) ; but Mr. Gatty, by a method differing from what follows, pointed out to the author that this idea was incorrect.

* It should be noted, however, that since the methods enabling the osmotic pressures to be calculated are independent of the specific volumes of the components in the solution (or mixed vapour), the presence of a third component does not necessarily affect the result.

Berkeley and Burton * obtain two equations, which, although derived from a consideration of solutions, are valid for mixed vapours ; these equations are

$$C_1S_1\frac{\delta P_1}{\delta C_2}=C_2S_2\frac{\delta P_2}{\delta C_1}; \qquad \frac{\partial P_1}{\partial C_2}=\frac{S_1}{u}\frac{\delta P_1}{\delta C_2}$$

$$\left(\text{and the analogous formula for } \frac{\partial P_2}{\partial C_1}\right).$$

From these equations, remembering that

$$\frac{\partial P_1}{\partial C_2}=-\frac{\partial P_1}{\partial C_1}$$

and using the new notation, it is readily seen that

$$\frac{\partial P_2}{\partial C_2}\Big/\frac{\partial P_2}{\partial C_1}=\frac{C_2\sigma_v}{C_1s_v},$$

where $P_1$ and $P_2$ are the two osmotic pressures of the mixed vapour.

But on the original assumption that $\pi_1=\pi_0$ and $\phi_1=\phi_0$, it is clear that $P_1=\phi_0$ ; $P_2=\pi_0$, hence

$$\frac{\partial \phi_0}{\partial C_2}\Big/\frac{\partial \pi_0}{\partial C_2}=\frac{C_2\sigma_v}{C_1s_v},$$

but the differentiation takes place with $\pi_1+\phi_1\,(=\pi_0+\phi_0)$ constant.

$$\therefore \qquad \frac{\partial \phi_0}{\partial C_2}-\frac{\partial \pi_0}{\partial C_1}=0.$$

On substituting, the condition required is

$$\frac{\partial \phi_0}{\partial C_2}\Big/\frac{\partial \pi_0}{\partial C_1}=\frac{C_2\sigma_v}{C_1s_v}=1. \quad . \quad . \quad . \quad . \qquad (1)$$

### *A General Method for Calculating Equilibrium Vapour Pressures.*

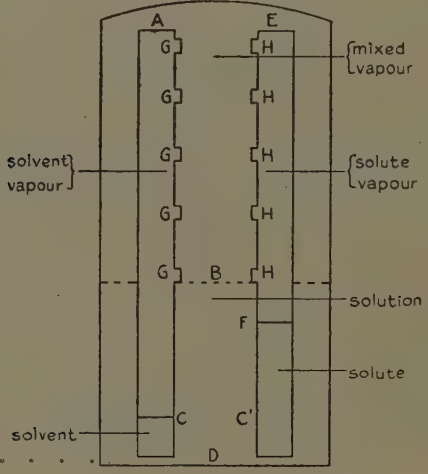
This method is applicable to all mixed vapours whether they conform to equation (1) or not ; and therefore is a means by which the osmotic pressures of any solution, made up of two volatile components, can be found.

Before proceeding with the analysis it seems advisable briefly to discuss the conditions envisaged.

* *Loc. cit.*, equations (14) and (18), p. 604.

Imagine a long closed tube containing a mixed vapour, stratified by gravity and diffusion, in equilibrium with a solution (at the bottom of the tube) the surface of which is at sea-level—no matter what be the concentrations at different levels in the solution or vapour, by withdrawing or adding one of the components to the vapour, a stratified mixed vapour can be obtained such that its bottom layer has a pressure $\pi_1+\phi_1$; and this layer will be in equilibrium with the top layer of the solution below it. If these two layers be cut off from the remainder

Fig. 1.




of the two phases the equilibrium will still obtain between them—the two layers are therefore the precise equivalent of a solution in equilibrium with its vapour when the former is under the pressure of its own vapour.

Consider the scheme shown in fig. 1, where the surface of the solution is at B, the sea-level. Both solution and the mixed vapour above it are stratified by gravity and diffusion *. Tubes A and E contain the pure solvent and

* Stratification by gravity and diffusion means that not only has gravity acted on molecules of different mass but osmotic attraction has brought these molecules into a steady state.

pure solute vapours (respectively) in contact with the pure liquids at the bottom of the tubes. The mixed vapour is in osmotic equilibrium with the solvent vapour through the membranes G, and with the solute vapour through the membranes H. Incidentally it may be pointed out that if there be osmotic equilibrium at any point there must be equilibrium at all other points, for otherwise perpetual motion would result.

Let the pressure of the mixed vapour just above B be $\pi_1+\phi_1$, a known quantity. Berkeley and Burton * give an equation connecting the pressure on a solution with the osmotic pressure. The reasoning by which this equation is derived is applicable to a mixed vapour stratified as in fig. 1, and, in the notation here used, becomes

$$(s_v-w_v)\,d(\pi_1+\phi_1)=s_v\,dP_1, \quad . \quad . \quad . \quad . \quad (2)$$

where $P_1$ is the osmotic pressure of the mixed vapour with reference to the solvent vapour. The corresponding equation for the solute considered as solvent is

$$(\sigma_v-w_v)\,d(\pi_1+\phi_1)=\sigma_v\,dP_2, \quad . \quad . \quad . \quad (3)$$

where $P_2$ is the complementary osmotic pressure—both osmotic pressures are measured against the pure vapours at the same heights above sea-level.

Take any † level K above B and integrate each equation between B and K, then

$$\int_{(\pi_1+\phi_1)_K}^{(\pi_1+\phi_1)_B}(s_v-w_v)\,d(\pi_1+\phi_1)=\int_{P_{1K}}^{P_{1B}}s_v\,dP_1, \quad . \quad . \quad . \quad (4)$$

$$\int_{(\pi_1+\phi_1)_K}^{(\pi_1+\phi_1)_B}(\sigma_v-w_v)\,d(\pi_1+\phi_1=\int_{P_{2K}}^{P_{2B}}\sigma_v\,dP_2. \quad . \quad . \quad . \quad (5)$$

Remembering that the osmotic pressure of the mixed vapour is the difference between the pressures on either side of the membrane, then integrating by parts and subtracting one equation from the other, the result is

$$\int_{(\pi_1+\phi_1)_K}^{(\pi_1+\phi_1)_B}s_v\,dp-\int_{(\pi_1+\phi_1)_K}^{(\pi_1+\phi_1)_B}\sigma_v\,dp=\int_{(\pi_1+\phi_1)_K-\pi_{0K}}^{(\pi_1+\phi_1)_B-\pi_{0B}}s_v\,dp-\int_{(\pi_1+\phi_1)_K-\phi_{0K}}^{(\pi_1+\phi_1)_B-\phi_{0B}}\sigma_v\,dp \quad . \quad . \quad (6)$$

* *Loc. cit.*, equation (6), p. 601.

† The distance between B and K should be large enough to give a substantial value to the left-hand integrals.

Although this equation contains five unknowns, namely, $(\pi_1+\phi_1)_K$, $\pi_{0B}$, $\pi_{0K}$, $\phi_{0B}$, and $\phi_{0K}$, it can be solved by trial and error as follows :—

Assume that the liquid solvent and solute are at definite depths below B and tabulate their vapour pressures for heights above their surfaces. Then take any value, less than $(\pi_1+\phi_1)_B$, for $(\pi_1+\phi_1)_K$, and calculate the left-hand side of (6). Search * the tables so as to find pressures that satisfy (6). The distance separating $\pi_{0B}$ from $\pi_{0K}$ must be the same as that separating $\phi_{0B}$ from $\phi_{0K}$, and remembering that $w_v=S_v$ at sea-level, the selected pressures must also, when inserted in (4) and (5), give identical values for $w_{vK}$. If it be borne in mind that $\pi_{0B}$ and $\phi_{0B}$ are the sea-level pressures, the positions of the pure liquids are determined and the height of K above B is found.

It may be pointed out that as the graphs of vapour pressure against heights end asymtotically on a vertical coordinate, the above-mentioned search will, in the majority of cases, be fairly easy.

A check is afforded by taking another value for $(\pi_1+\phi_1)_K$ and repeating the operation.

Incidentally, it should be mentioned that as there is no limit to the distance between B and K the laws connecting the pressures and specific volumes of the pure vapours must be such that they satisfy this condition.

If while solving equation (6) it is found that

$$(\pi_1+\phi_1)_B=\pi_{0B}+\phi_{0B} \quad \text{and} \quad (\pi_1+\phi_1)_K=\pi_{0K}+\phi_{0K},$$

then it is probable that the mixed vapour conforms to equation (1), and its partial pressures will be

$$\pi_{1B}=\pi_{0B}\,; \quad \phi_{1B}=\phi_{0B}.$$

This assumption will be confirmed if (4) and (5) give the same values for $w_{vk}$.

*Calculation of the Concentration.*

In connexion with the foregoing it may be useful to show that even when the partial pressures are unknown, the concentrations, at any height, of the components can be evaluated. If required $s_1$ and $\sigma_1$ could also be found.

* Obviously the search will be facilitated if a series of solutions are being dealt with.

Let $C_1$ and $C_2$ be the concentrations of the mixed vapour at sea-level. Then

$$s_1 = w_v - C_2 \frac{\partial w}{\partial C_2} \quad \text{and} \quad \sigma_1 = w_v - C_1 \frac{\partial w}{\partial C_1}\text{*}, \quad . \cdot \quad (7)$$

Substituting these in equation (29) of Berkeley and Burton's paper † (applied to both osmotic pressures), and remembering that

$$\frac{\partial w}{\partial C_1} = - \frac{\partial w}{\partial C_2},$$

it is readily deduced that

$$\left\{\left(1 - \frac{\partial P_1}{\partial p}\right) s_v - w_v\right\} \Big/ \left\{\left(1 - \frac{\partial P_2}{\partial p}\right) \sigma_v - w_v\right\} = - \frac{C_2}{1 - C_2}, \quad . \quad . \quad . \quad (8)$$

$\frac{\partial P_1}{\partial p}$ and $\frac{\partial P_2}{\partial p}$, the changes in the osmotic pressures when the concentrations are kept constant, can be found by a method (independent of $s_1$ and $\sigma_1$), which is detailed on p. 494; and as $w_v$, $s_v$, and $\sigma_v$ are known, $C_2$ can be calculated for different heights.

Although this calculation may be somewhat lengthy yet as it leads to a more complete knowledge of the stratification it may be of use in other researches ‡.

* Callendar, Proc. Roy. Soc. series A, vol. lxxx. p. 470.

† *Loc. cit.* p. 616. $\frac{\partial P_1}{\partial p} = \frac{s_v - s_1}{s_v}$ and $\frac{\partial P_2}{\partial p} = \frac{\sigma_v - \sigma_1}{\sigma_v}$.

‡ For instance, a plausible guess at the magnitude of the partial pressures can be made for mixtures in which $(\pi_0 + \phi_0) - (\pi_1 + \phi_1)$ never becomes too great. For $s_0$ and $\sigma_0$ can be calculated for any concentration and any pressure (from Callendar's equation and B and B's (29)). So starting with a "solvent" dilute mixed vapour where $\pi_1$ can be put equal to $\pi_0$ (therefore $\phi_1$ is known) and ending with a "solute" dilute vapour where similarly $\phi_1$ and $\pi_1$ are known, a connexion between $\pi_1$ and $s_\nu$ (and $\phi_1$ and $\sigma_\nu$) can be postulated so as to account for the differences between $\pi_1 + \phi_1$ and $\pi_0 + \phi_0$ in the intermediate mixtures. Also assuming that there are no convection currents, the steady state which the components of the atmosphere would reach under the action of gravity and osmotic diffusion might be calculated (possibly it may even throw light on the movement of the various strata that reflect radio waves) when the connexion between osmotic pressure and temperature has been solved.

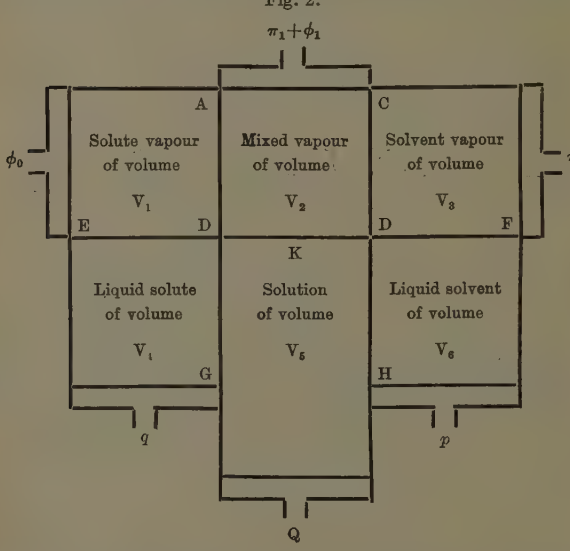
*An Equation connecting the Vapour Pressures.*

Consider the system shown in fig. 2. AB, BE, BG, and BK are solute membranes *, while CD, DF, DH, and DK are solvent membranes.

The solution is under its own vapour pressure $(\pi_1+\phi_1)$.

The solute and solvent are under negative pressures $q$ and $p$, and their vapour pressures $\phi_0$ and $\pi_0$ are supposed

Fig. 2.




to be in osmotic equilibrium with $\phi_1$ and $\pi_1$ (respectively), the corresponding partial pressures in the mixed vapour.

It should be explained that the three liquids under line EF are not necessary for the ensuing cycle, but will be required later on.

Perform the following stages of an isothermal thermodynamic cycle :—

*Stage* (1).

Close EB, BK, KD, and DF, and pass one gramme of solute vapour from $V_2$ to $V_1$ and simultaneously $C_v$

* The locution "solute membrane" means a membrane permeable only to the substance designated.

grammes of solvent vapour from $V_2$ to $V_3$—the work done on the system is

$$-(\pi_1+\phi_1)(1+C_v)S_v+\phi_0\sigma_v+\pi_0 C_v s_v.$$

*Stage* (2).

Reduce the pressure $(\pi_1+\phi_1)$ to $(\pi_1'+\phi_1')$; $\phi_0$ to $\phi_0'$; $\pi_0$ to $\pi_0'$, maintaining equilibrium across the membranes while doing so—the work done is

$$\int_{\pi_1+\phi_1}^{\pi_1'+\phi_1'}(\pi_1+\phi_1)\,d\{V_2-(1+C_v)S_v\}+\int_{\phi_0}^{\phi_0'}\phi_0 d(V_1+\sigma_v)+\int_{\pi_0}^{\pi_0'}\pi_0 d(V_3+C_v s_v).$$

*Stage* (3).

Pass one gramme of solute vapour from $V_1$ to $V_2$ and simultaneously $C_v$ grammes solvent vapour from $V_3$ to $V_2$—the work done is

$$(\pi_1'+\phi_1')(1+C_v)S_v'-\phi_0'\sigma_v'-\pi_0' C_v s_v'.$$

*Stage* (4).

Restore original pressures and then open the membranes E B, B K, K D, and D F—the work done is

$$\int_{\pi_1'+\phi_1'}^{\pi_1+\phi_1}(\pi_1'+\phi_1')\,dV_2+\int_{\phi_0'}^{\phi_0}\phi_0'\,dV_1+\int_{\pi_0'}^{\pi_0}\pi_0'\,dV_3.$$

Summing up, integrating by parts, and equating to zero, the result is

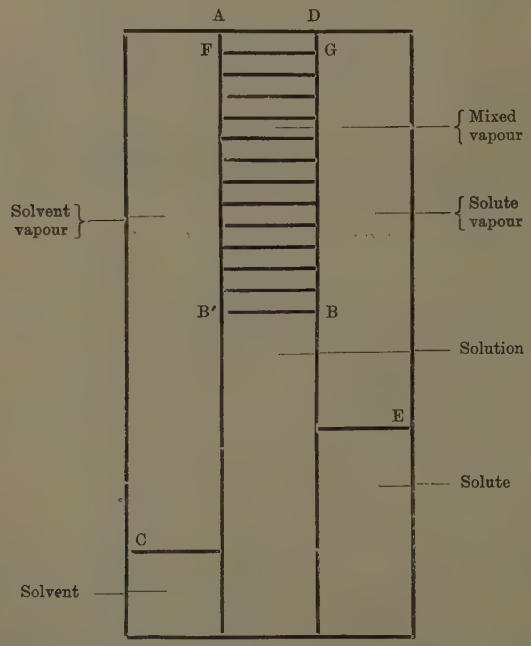
$$(1+C_v)\int_{\pi_1+\phi_1}^{\pi_1'+\phi_1'}S_v\,dp=\int_{\phi_0}^{\phi_0'}\sigma_v\,dp+C_v\int_{\pi_0}^{\pi_0'}s_v\,dp\ *. \quad . \quad (9)$$

To use equation (9) for the purpose indicated on p. 489 $\left(i.\,e.,\text{ to determine }\frac{\partial P_1}{\partial p}\text{ and }\frac{\partial P_2}{\partial p}\right)$, consider fig. 3, where there are three tubes (AC containing the solvent vapour, AB the mixed vapour, and DE the solute vapour) alongside one another. The surface of the solution is at sea-level, B′B, and the mixed vapour above it is stratified by gravity only.

* This equation can be obtained by integrating Berkeley and Burton's equation (29) (applied to both vapour components) and combining the result with $w_v=C_1 s_1+C_2\sigma_1$. Incidentally it may throw light on the constants of Van der Waals's equation when applied to the mixed vapour.

This latter statement should be explained. The contents of AC, for instance, when in a steady state, can be said to be stratified by gravity because the specific volume of the vapour changes with height above the pure liquid and is dependent on the gravity potential. The same phrase can be applied to the mixed vapour provided a steady state, other than that postulated as

Fig. 3.




the condition in fig. 1, can be conceived. That is provided the action of gravity on molecules differing in mass can be stopped.

This can be done in the following way. First of all consider only the two tubes AC and AB. Let B′B be a solvent membrane. Place a great number of infinitely thin, weightless, piston membranes across AB that are permeable to the solvent only—these membranes, if the

mixed vapour was originally gravity stratified, stop any molecular displacements, for it is easy to show that otherwise perpetual motion would result.

Thus, close membrane B′B and open a solvent membrane in the tube's side just above B′. If the level of C is such that the pure solvent vapour at B′ is in osmotic equilibrium with the mixed vapour on the other side, then no movement will take place across this membrane. Now open another solvent vapour membrane close to A. There must be osmotic equilibrium across it ; for if not, assume that the pressure inside AB is the greater, then solvent vapour passes out of AB through the membrane at A into AC and, if the surface of the solvent liquid is shut off, vapour will pass through the membrane at B′ into tube AB, so perpetual motion takes place.

Hence the osmotic pressure of the mixed vapour is given for any height from B′B upwards by the difference between the pressure inside AB and that in AC. To obtain the complementary osmotic pressure, remove tube AC and also the solvent membranes in AB. Replace the latter by solute membranes and connect up with the solute in DE. The preceding argument will then apply *.

Consequently the three tubes can be used as a scheme showing a mixed vapour, stratified by gravity only (thus having the same concentration throughout), in osmotic equilibrium with the pure vapours outside it.

For any mixed vapour its two osmotic pressures can be calculated by means of (9) as follows :—

In the discussion on p. 488 it was shown that the pressure of a mixed vapour together with the pressure of the pure vapours in equilibrium with it could be calculated for any height K above B in fig. 1. Assume that this has been done and the pressures are $(\pi_1+\phi_1)_K$, $\pi_{0K}$, and $\phi_{0K}$ ; let these pressures now designated as $\pi_1+\phi_1$, $\pi_0$, and $\phi_0$ be those at level B′B in fig. 3—then the unknowns in (9) are $\pi_1'+\phi_1'$, $\pi_0'$ and $\phi_0'$ ; but $\pi_1'+\phi_1'$ can

* There is another way, which, briefly stated, is to fix FG and remove the other pistons. Then closing all solute membranes, fill AG with mixed vapour of the same concentration as that just above B′B and having a pressure such that when the contents of FB are replaced by solvent vapour there will be equilibrium across FG and B′B. By closing all the solvent membranes and refilling FB with gravity stratified solute vapours in equilibrium across B′B, it can be shown that perpetual motion will take place unless there now is equilibrium across FG. Repeating the operations for different heights the mental aspect of a mixed vapour stratified by gravity only is obtained.

be calculated for any height above B′B and $\pi_0'$ and $\phi_0'$ can be found, as they must be equal distances from $\pi_0$ and $\phi_0$. Hence $P_1(=\pi_1'+\phi_1'-\pi_0')$ and $P_2(=\pi_1'+\phi_1'-\phi_0')$ are found for the pressure $p(=\pi_1'+\phi_1')$ of the mixed vapours. Then repeating this for a sufficient number of heights, $\frac{\partial P_1}{\partial p}$ and $\frac{\partial P_2}{\partial p}$ for the mixed vapour at height K in fig. 1 can be found from graphs; and the concentration, as already shown, at that height can be derived.

By comparing the mixed vapours (stratified by gravity and diffusion) in fig. 1 to that (stratified by gravity only) in fig. 3 it is evident that the concentration just above B is the same in both cases—so also are both osmotic pressures (direct and complementary). In fig. 1, for any height above B, the concentration and pressure can, as shown, be calculated. In fig. 3 both these factors are known. Since in both cases the osmotic pressures at the same height are measured against the same pressure of solvent vapour (or solute vapour) this gives a direct means of connecting the osmotic pressures with the concentrations.

Should the mixed vapour conform to equation (1) its osmotic pressures can be calculated directly by means of (9) without having to evaluate $\pi_0$ and $\phi_0$ as on p. 488. For in this equation the unknowns now are $\pi_0, \pi_0', \phi_0, \phi_0'$, and $\pi_1+\phi_1=\pi_0+\phi_0$, also $\pi_1'+\phi_1'=\pi_0'+\phi_0'$, so that a search in the tables of vapour pressures will readily give the required values to satisfy the equation.

The foregoing is a way of obtaining the values of $\frac{\partial P_1}{\partial p}$ and $\frac{\partial P_2}{\partial p}$ by means of (9) derived in the manner stated (*i. e.*, pressures reduced so no condensation is possible), but for general use it is advisable to show that (9) can be obtained even when the pressures are increased.

*A General Equation connecting the Vapour Pressures.*

Before giving the analysis it is necessary to discuss what takes place when the pressure on a solution, in equilibrium with its mixed vapour, is raised, and may thereby cause an alteration in the concentrations. It is

evident that the change in pressure on the solution and the corresponding change in the pressure of the mixed vapour will cause a flow of one component from the solution to the mixed vapour above it (they are supposed to be separated from one another by a membrane permeable to both components) and then there must be a counter flow of the other component in the reverse direction. In other words, unless each component can move there will be no movement at all. Incidentally, it may be mentioned that a decrease in the pressure on the solution and its mixed vapour may also cause changes in concentrations.

Referring again to fig. 2. With AB, BG, CD, DH, and DK shut, and BK open, change the pressures on solution and mixed vapour in such a way that there is no flow through the membrane BK—obviously as there cannot be any counter flow there will be no changes in the concentrations—the question of condensation will be discussed later.

Now suppose these pressure changes had taken place with BE and BG open, the pressures on $V_1$ and $V_4$ being changed so that equilibrium was maintained across BE and BG, then there must be equilibrium across AB when it is opened ; for, if not, it is easy to show that perpetual motion of the solute would occur.

A thermodynamic cycle can now be carried out, but it will have to be performed in a slightly different way to that described on p. 491.

*Stage* (1)

With all membranes open, force $C_v$ grammes of solvent from $V_2$ into $V_3$ and simultaneously one gramme of solute from $V_2$ into $V_1$. Work done is

$$-(\pi_1+\phi_1)(1+C_v)S_v+\pi_0 C_v s_v+\phi_0\sigma_v.$$

*Stage* (2).

Close membranes AB and BK and increase the pressures to $Q'$, $\pi_1'+\phi_1'$, $\pi_0'$, $\phi_0'$, $p'$, and $q'$, while maintaining osmotic equilibrium across BG, DH, and CD, and vapour equilibrium across EB, KD, and DF. As already shown, there would be vapour equilibrium across AB and BK

should these be opened and KD and DC closed. Work done is

$$\int_{Q}^{Q'} Q\, dV_5 + \int_{p}^{p'} p\, dV_6 + \int_{q}^{q'} q\, dV_4 + \int_{\pi_0}^{\pi_0'} \pi_0\, d(V_3 + C_v s_v) + \int_{\phi_0}^{\phi_0'} \phi_0\, d(V_1 + \sigma_v) + \int_{\pi_1+\phi_1}^{\pi_1'+\phi_1'} (\pi_1 + \phi_1\, d\{V_2 - (1+C_v)S_v\}.$$

*Stage* (3).

Return into $V_2$ simultaneously $C_v$ grammes of solvent from $V_3$ and one gramme of solute from $V_1$. This movement of the two component vapours cannot be done in the ordinary manner because it might cause changes in the two concentrations; but it can be done thus :—

Let the volume of $V_1$ be nearly infinitely great; then, with CD and KD shut, while BK and AB are open, return a small proportion of the one gramme of solute; then close BK and AB, and open CD and KD and return a small part of $C_v$ grammes of solvent. Repeat these operations until one gramme of solute and $C_v$ grammes of solvent have been moved over. The work done is

$$(\pi_1' + \phi_1')(1+C_v)S_v' - \pi_0' C_v s_v' - \phi_0' \sigma_v.$$

*Stage* (4).

Restore original pressures. If $V_1$, $V_3$, $V_4$, and $V_6$ be shut off their pressures can be restored directly, but for $V_2$ and $V_5$ either BK or DK must be open and the pressures regained while keeping equilibrium across whichever membrane is functioning. Work done is

$$\int_{Q'}^{Q} Q'\, dV_3 + \int_{p'}^{p} p'\, dV_6 + \int_{q'}^{q} q'\, dV_0 + \int_{\pi_0'}^{\pi_0} \pi_0'\, dV_3 + \int_{\phi_0'}^{\phi} \phi_0'\, dV_1 + \int_{\pi_1'+\phi_1'}^{\pi_1+\phi_1} (\pi_1' + \phi_1')\, dV_2.$$

Summing up and equating to zero, the result is equation (9).

A way of showing that no condensation in the mixed vapour takes place when the pressures are increased is to perform the operation in the following manner. Close BK and KD and increase the pressures to $\pi_1' + \phi_1'$, $\phi_0'$, $\pi_0'$, $p'$, $q'$, and Q′ maintaining equilibrium across all the open membranes. Now close CD and open BK—it will

be apparent that there must be equilibrium across BK for, as shown already, perpetual motion would take place—similarly for the solvent side—and at the same time no condensation takes place because there is nothing to start it doing so.

Obviously a similar cycle to the foregoing, but now straightforward, can be performed on the contents of $V_4$, $V_5$, and $V_6$ when all the membranes on the line EF are closed. The result will be

$$(1+C_z)\int_Q^{Q'} \bar{S}_z\,dp = \int_q^{q'} \sigma_z\,dp + C_z \int_p^{p'} s_z\,dp. \quad . \quad (10)$$

On substituting Prof. Porter's two equations,

$$\int_q^{g'} \sigma_z\,dp = \int_{\phi_0'}^{\phi} \sigma_v\,dp \quad \text{and} \quad \int_p^{p'} s_z\,dp = \int_{\pi_0'}^{\pi_0} s_v\,dp,$$

the result is

$$(1+C_z)\int_Q^{Q'} \bar{S}_z\,dp = \int_{\phi_0'}^{\phi_0} \sigma_v\,dp + C_z \int_{\pi_0'}^{\pi_0} s_v\,dp. \quad . \quad . \quad (11)$$

Then subtracting (9) from (11),

$$(1+C_z)\int_Q^{Q_1'} \bar{S}_z\,dp = (1+C_v)\int_{\pi_1'+\phi_1'}^{\pi_1+\phi_1} \bar{S}_v\,dp + (C_z-C_v)\int_{\pi_0'}^{\pi_0} s_v\,dp. \quad (12)$$

(This equation can also be derived by a suitable thermodynamic cycle.)

Obviously (12) can be put in the form :

$$(1+C_z)\int_Q^{Q'} \bar{S}_z\,dp = (a+C_z)\int_{\pi_1\ \phi_1'}^{\pi_1+\phi_1} \bar{S}_v\,dp + (1-a)\int_{\phi_0'}^{\phi_0} \sigma_v\,dp, \quad (13)$$

where $C_z = aC_v$.

### *Methods for determining the Osmotic Pressures of the Solution.*

Two methods which are independent of the specific volumes of the solute and solvent *in* the solution will be described. The first method is based on equation (11) and does not require any knowledge of the compressibility of the solution unless very accurate results are wanted

By putting $Q=\pi_1+\phi_1$ three quantities in (11) are known, viz., Q, $\pi_0$, and $\phi_1$. Let Q′ be the pressure to be put on the solution such that the pressure of the mixed vapour is $\pi'+\phi_1'$, and this vapour is in osmotic equilibrium

with the vapour of the solute at pressure $\phi_{00}$ (*i. e.*, when the solute is under the pressure of its own vapour only). The unknowns now are Q′ and $\pi_0'$. To find $\pi_0'$ examine fig. 1 ; the distance BF between sea-level and the surface of the solute is known, so, on referring to fig. 3, BE will be known.

Evidently then by placing (in fig. 3) the solution at level E and applying the pressure Q′ to it, its mixed vapour will have a pressure $\pi_1'+\phi_1'$ and will be in equilibrium with both the pure vapours on either side of it and the pressure of the solvent $\pi_0'$ can be calculated. These values inserted in (11) give the osmotic pressure $P_2=Q'-\phi_0$.

A value for $P_1$ (the other osmotic pressure) can be found. Let Q′ be the pressure on the solution necessary to put its vapour in osmotic equilibrium with $\pi_{00}$. Place the liquid solute, whose surface is now separated from its vapour by a membrane, at level C ; and subject it to a pressure such that the resulting vapour pressure at level B is the same as before, namely, $\phi_0$. Its vapour pressure at level C will be $\phi_0'$, the value required ; and this value can be calculated because the rate of change in vapour pressure with depth below B is known and BC is known. Hence, putting these quantities in (11) the osmotic pressure $P_1=Q'-\pi_{00}$ is found.

It must be remembered that these two osmotic pressures are not truly complementary ; they have been derived when the solution is under different pressures. In most cases, as liquids are but slightly compressible, the two are a close approximation to the correct values. If, however, greater accuracy be required it can be obtained for one pair of osmotic pressures in the following way.

Referring again to fig. 3 *. Assume that the compressibilities of the three liquids have been determined experimentally, and that the solution itself is gravity stratified and this stratification is made permanent by the use of many infinitely thin membranes. Then the two values of Q′ obtained in the preceding analysis will give the levels at which the solution is subjected to these

* The following discussion is merely a modification of that used by Berkeley and Hartley in the paper where they showed how the connexion between the vapour pressure of the solvent and the osmotic pressure of the solution (the solution having but one volatile constituent) could be derived—they, however, omitted to consider the compressibilities.

pressures. It will readily be seen that the pressure of the solute at the same level as the free surface of the solvent can be calculated; and at this level the solute must be in osmotic equilibrium with the solution, thus giving the correct complementary osmotic pressure.

Obviously, in fig. 3, with the solution gravity stratified as mentioned, and knowing the necessary compressibilities and vapour pressures, a calculation of the two hydrostatic balances at level C will give the two osmotic pressures required without recourse to equation (11).

Incidentally, it may be pointed out that (12) and (13) give the values of $\pi_1' + \phi_1'$ which are the result of increasing the pressure on the solution from $\pi_1 + \phi_1$ to either of the two calculated values of Q′. As Q′ may be of considerable magnitude, it may be useful in researches on the "pressure-concentration" hypothesis discussed in the Appendix.

*Another method is as follows*:—It gives an approximate value for the osmotic pressure of the solution at B in fig. 1.

The positions of the surfaces of the liquids at C and F have been found by the method already given (p. 488), and are such that the vapour pressures at B are in equilibrium with the partial pressures there. Therefore, if suitable membranes be opened anywhere below C and F, osmotic equilibrium will obtain across them; in fact, the situation is precisely that shown in Berkeley and Burton's fig. 1 *. Hence equation (6) of that paper is again applicable.

In the new notation, this equation, applied to the two osmotic pressures gives

$$(s_z - w_z)\,dp = s_z\,dP_1 \quad \text{and} \quad (\sigma_z - w_z)\,dp = \sigma_z\,dP_2, \quad (14)$$

where $P_1$ and $P_2$ now are the osmotic pressures of the solution referred to the solvent and the solute respectively.

Let C′ be at the same level as C. Then, taking D as a known distance below C, the equations can be integrated between C and D, giving

$$\int_{\pi_1+\phi_1+L}^{\pi_1+\phi+K} (s_z - w_z)\,dp = \int_{P_{1D}}^{P_{1C}} s_z\,dp \quad \text{and} \quad \int_{\pi_1+\phi_1+L}^{\pi_1+\phi_1+K} (\sigma_z - w_z)\,dp = \int_{P_{2D}}^{P_{2C}} \sigma_z\,dp$$

* *Loc. cit.* p. 599.

where K represents the pressure of BC of solution and L that of BD ; but

$P_{1C}=\pi_1+\phi_1+K-\pi_{0C}$ ; $P_{1D}=\pi_1+\phi_1+L-M-\pi_{0C}$ ; and

$P_{2C}=\pi_1+\phi_1+K-\phi_{0C}-N$ ; $P_{2D}=\pi_1+\phi_1+L-\phi_{0C}-(N+O)$,

where M, N, O represent the pressures of the pure liquids.

Substituting these values the equations become

$$\int_{\pi_1+\phi_1+L}^{\pi_1+\phi_1+K}(s_z-w_z)\,dp=\int_{\pi_1+\phi_1+L-\pi_{0C}-M}^{\pi_1+\phi_1+K-\pi_{0C}} s_z\,dp \quad . \quad . \quad . \quad (15)$$

$$\int_{\pi_1+\phi_1+L}^{\pi_1+\phi_1+K}(\sigma_z-w_z)\,dp=\int_{\pi_1+\phi_1+L-\phi_{0C}-(N+O)}^{\pi_1+\phi_1+K-\phi_{0C}-N} \sigma_z\,dp \quad . \quad . \quad . \quad (16)$$

Proceeding as before so as to eliminate $w_z$, the result is

$$\int_{\pi_1+\phi_1+L}^{\pi_1+\phi_1+K} s_z\,dp-\int_{\pi_1+\phi_1+L}^{\pi_1+\phi_1+K}\sigma_z\,dp=\int_{\pi_1+\phi_1+L-\pi_{0C}-}^{\pi_1+\phi_1+K-\pi_{0C}} s_z\,dp-\int_{\pi_1\phi_1+L-\phi_{0C}-(N+O)}^{\pi_1+\phi_1+K-\phi_{0C}-N}\sigma_z\,dp \quad . \quad (17)$$

In this equation $(\pi_1+\phi_1)$ and $\pi_{0C}$ are known ; and $\phi_{0C}$ can be calculated.

Since the specific gravities and compressibilities of the pure liquids have been determined the pressures M, N, and O, also are known—thus the only unknowns are K and L. By trial and error values of K and L can be found to satisfy (17), and at the same time these when inserted in (15) and (16) must give the same value for

$$\int_{\pi_1+\phi_1+L}^{\pi_1+\ +K} w_z\,dp \ .$$

The two osmotic pressures of the solution at levels C and D are thus found, but it is required to know its value at level B—this can be done by repeating the process for other levels and extrapolating back to B. Without going into details, it will be evident that this extrapolation can be tested by suitable modifications of (15) and (16) ; for as $w_z=s_z$ at level B, therefore each equation should give the same value for $w_z$ at any level below B.

If the concentration of the solution at any level below B be required it can be calculated by the method given by Berkeley and Burton *. Another way, independent of $s_1$ and $\sigma_1$, can be devised ; but for great accuracy it is

* *Loc. cit.* Appendix 2.

long and laborious. The factors required are $P_1$, $P_2$, and $w_z$, determined by a method similar to that given on p. 498, for a number of solutions of different concentrations and for a series of pressures, $p$, applied to each solution. Then tabulating these factors, for each solution, against $p$, it will readily be seen that any values of $P_1$, $P_2$, $w_z$, and $p$, calculated for a particular level in fig. 1, can be found in one of the tables. The concentration of the solution, the properties of which are given in this table must, in general, be the concentration of the solution at the level under discussion.

It should be mentioned that this method has been detailed, not because it is a convenient way of calculating osmotic pressures, but because it throws light on the osmotic pressures and concentrations of the solutions at levels below B. This, as will be seen in the Appendix, will help to elucidate the "pressure-concentration" hypothesis.

Another reason for giving the details is that hitherto the osmotic pressures of solutions, which vary in concentration, but contain only one volatile constituent, have been compared * when the pressure on the solvent is constant (the reason being that the experiments themselves were carried out under this condition). Theoretically, the osmotic pressures could be compared when the solutions are under a constant pressure; and then it might be advisable to take as standard the pressure on the saturated solution that is required for osmotic equilibrium with the solvent when the latter is under the pressure of its own vapour only.

With two volatile components, it is probable that more information will accrue if the two true complementary osmotic pressures are available, and it may be that the solutions should then be compared when they are under the pressures of their own mixed vapours only. If, however, they have to be under a constant pressure, this standard pressure will be hard to find; so it may be useful to point out that the search is facilitated by the schemes shown in figs. 1 and 3.

The author is glad to have the opportunity of thanking Sir Harold Hartley and Mr. O. Gatty for their helpful advice.

* For instance, see "An Empirical Equation for Calculating Osmotic Pressures," Berkeley, Phil. Mag. xvi. p. 298 (1935).

## Appendix.

The change, if any, in the concentration of a solution and its mixed vapour when the pressure on the solution is altered may be found by experiments carried out on the vapour pressure of a volatile liquid which freely dissolves a chemically inert gas. Air cannot be used, as it is uncertain that there will not be selective absorption of one of the component gases.

The main idea is to pass a measured quantity of the gas, under definite and constant pressures, over the liquid, and compare the latter's loss of weight when the pressure is high to its loss when the pressure is low.

Broadly speaking, the method is to saturate the liquid (which now becomes the solution) with gas, say, at the high pressure and weigh the solution. Then pass the gas, at the same high pressure, over the solution and note the latter's loss of weight ; this loss gives the pressure of the mixed vapour.

To determine the partial pressure there are three cases to be considered :—

(*a*) Where the vapour component of the mixed vapour can be absorbed quantitatively by chemical or other means ; this case is obvious.

(*b*) Where the vapour cannot be absorbed, but the gas conforms to equation (1) ; here the partial pressure of the gas in the mixed vapour equals the pressure at which it flows over the solution, and therefore the two partial pressures together with the concentration in the mixed vapour are found.

(*c*) Where the gas does not conform to (1), nor can the vapour be absorbed. Under these conditions the procedure is somewhat more elaborate. Saturate the solution with gas at the lower pressure and then pass gas at the higher pressure through the vessel for a time slightly more than sufficient to allow equilibrium to obtain. Repeat the experiment for a considerably longer time. The loss of weight per unit time in the two experiments will be different (it is assumed that the speed and pressure of the gas are exactly the same in the two experiments—other precautions and details will readily suggest themselves). This difference depends on the fact that during the first part of each experiment the solution will be dissolving

gas, and as this absorption causes a diminution in the partial pressure of the vapour, there may also be a loss of vapour. The sum of these two effects can be calculated from the unit time loss.

Now carry out somewhat similar experiments with the solution saturated at the high pressure and the gas passing at the low. The unit time losses will again be different, but the difference will now be due only to the loss of weight of some dissolved gas, there being no simultaneous loss of vapour. Hence the amount of gas moved can be calculated.

For an approximate solution of the problem it can be assumed that the amount of gas dissolved or released in the two sets of experiments is the same, then the known factors are : the pressures of the mixed vapours when the solution is under the two gas pressures, the quantity of gas absorbed, and the loss of vapour.

It will readily be seen that these four quantities will determine the four partial pressures. But should a more accurate result be desired, it can be obtained by repeating the two sets of experiments with different quantities of solution. This will give a means of calculating the amount of gas absorbed or released to a closer approach to the truth.

The concentrations of the two solutions can be determined by chemical analysis.

Another method that will give some information on this subject is obtainable from the scheme shown in fig. 1. Consider a layer of solution at some definite level, R, between B and F. On opening suitable membranes at R in the tubes AC and EF it is obvious that osmotic equilibrium will exist between the solution and the two pure vapours ; evidently this equilibrium can only be that between the mixed vapour and the other vapours.

Since $\pi_{0R}$ and $\phi_{0R}$ are known, it is possible by trial and error to find the value of $(\pi_1+\phi_1)_R$ by means of (6) integrated between B and R. The selected value can be confirmed by seeing whether it gives, when applied to (4) and (5), the same value for $w_{vR}$ Then the concentration $C_z$ at R can be found by the method already detailed (p. 489) ; and with these constants, together with a knowledge of the necessary compressibilities, the pressure, Q, on the solution at R is derivable.

Let the solution in fig. 3 have the concentration $C_z$,

and, if the level R in fig. 1 has been chosen so that the solution in fig. 3 at some level, R′, between B and E is under a pressure Q, the two mixed vapours, one at R, in fig. 1, the other at R′ in fig. 3, can be compared.

All the factors for the mixed vapour of the solution at R′ (fig. 3) are known ; and for the mixed vapour at R (fig. 1) $C_{1R}$ (or $C_{2R}$) can be found by plotting the known values above B against either $\pi_0$ or $\phi_0$ and extrapolating down to R (incidentally it may be mentioned that, if desired, a more accurate method is available).

Thus for certain special cases the " pressure-concentration " hypothesis can be tested.

There seems to be yet another and apparently a simple method of testing this phenomenon ; it depends on how sensitive the changes in the concentration are to a change in pressure. If very sensitive, then tests under different gravitational forces due either to changes in altitude or changes in latitude might be sufficient.

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## XXXIV. *Wave-length Measurements of the* NiL$\alpha_{1,2}$ *and* NiL$\beta_1$ *Lines of Metallic Nickel and of Nickel Salts.*

*By* J. Shearer, *B.A., M.Sc., Lecturer in Physics, University of Western Australia* *.

A CHANGE of wave-length has been observed in this laboratory † for emission lines of elements in chemical combination, relative to the same lines of the pure element. The effect has been observed in emission lines which belong to the K-series and in emission lines which belong to the L-series.

Prof. Siegbahn suggested to the writer an investigation of the L-spectrum of nickel and of nickel salts, with a view to discovering whether any effects due to chemical combination may be observed in the L-series of this element. The L$\alpha_{1,2}$ and L$\beta_1$ lines were selected for investigation. Thoraeus ‡ observed a structure in the

* Communicated by Prof. Manne Siegbahn, D.Sc.

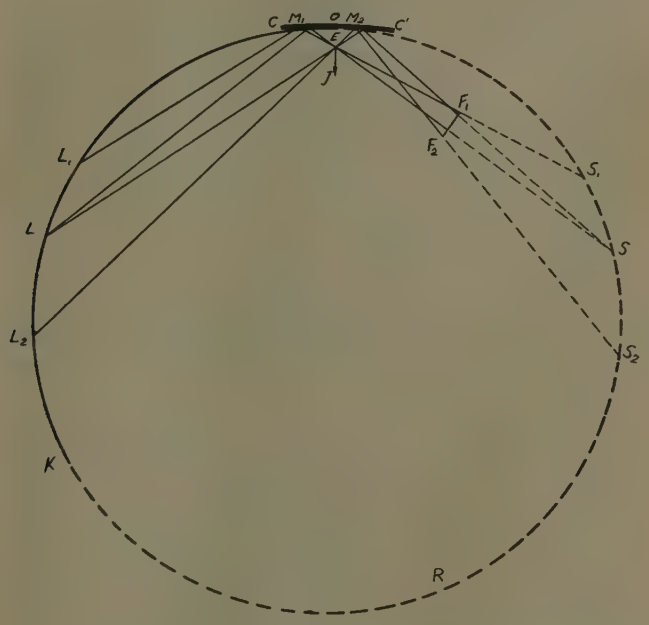
† *E.g.*, Ray, Phil. Mag. xlix. p. 168 (1925), and l. p. 505 (1925); Bäcklin, *Zeit. für Phys.* xxxiii. p. 547 (1925), and xxxviii. p. 215 (1925); Thoraeus, Phil. Mag. ii. p. 1007 (1926); Siegbahn and Magnusson, *Zeit. für Phys.* lxxxvii. p. 291 (1934); Karlsson and Siegbahn, *ibid.* lxxxviii. p. 76 (1934).

‡ Thoraeus, *loc. cit.*

$NiL\alpha_{1,2}$ line of trivalent nickel oxide, which did not appear with the metal or with the bivalent nickel sulphide.

The method employed in the present investigation was that of the concave cylindrical crystal used as a reflexion grating. The principle of the method is illustrated diagrammatically in fig. 1.

Fig. 1.




CC′ is the crystal whose centre O lies on the Rowland circle R. The radius of the crystal is equal to the diameter of the Rowland circle. The kasette and crystal are mounted on the same base. The boundary of the kasette (along which the photographic film lies) may be considered (for diagrammatic purposes) to extend from O to K along the Rowland circle, as shown. $F_1F_2$ is the focal spot on the anticathode. J is an auxiliary jaw (fixed to the crystal-holder) the distance of which from

the crystal may be adjusted. It serves the purpose of controlling the effective aperture of the crystal.

L is the point at which are focussed those rays from the focal spot whose grazing angle of incidence on the crystal is equal to the angle SOC′ (that is, SO=OL). It is apparent that the total range of grazing angles that may be covered with a given setting of crystal, jaw, and kasette, relative to one another and to the centre of the focal spot, is controlled by the horizontal length ($F_1F_2$) of the focal spot. The total wave-length (or $n\lambda$) range is that corresponding to the distance $L_1L_2$ along the Rowland circle, where $L_1$ and $L_2$ are defined by the facts that $L_1M_1$ is the reflected ray whose incident ray is $F_1EM_1$ (that is $S_1O=OL_1$), and $L_2EM_2$ is the reflected ray whose incident ray is $F_2M_2$ (that is, $S_2O=OL_2$). The focal spot is therefore equivalent to a source $S_1S_2$ on the Rowland circle, the intensity of which is vanishingly small in the neighbourhood of $S_1$ and $S_2$.

In order conveniently to vary the range of grazing angles obtainable with one setting, kasette and crystal (and therefore the Rowland circle) may be rotated about O, the anticathode (that is $F_1F_2$) remaining fixed. Clockwise rotation clearly gives grazing angles less than those shown in the figure, and counter-clockwise rotation, greater.

The apparatus employed was designed on the above principle. Such a spectrometer has already been described *. In the present work the adjustments were carried out along the lines there indicated. The radius of curvature of the kasette is 250 mm. The radius of curvature of the crystal-holder to which the crystal is fitted is 500 mm. Optical tests are necessary to verify that the crystal assumes this radius. A mica crystal 0·5 mm. thick was used. As the grazing angles for the $NiL\alpha_{1,2}$ and $NiL\beta_1$ lines are in the region of 46 degrees with mica, a kasette designed for grazing angles of 16 to 60 degrees was used. The clearance of the jaw J from the crystal (that is, the distance EO) was adjusted to 5 mm. This gave a crystal aperture of 10 mm. for a 45 degree grazing angle. A hole in the tube surrounding the anticathode, through which the X-ray beam passes, controls the effective size of the focal spot. One thickness

* Karlsson and Siegbahn, *loc. cit.*

of 0·5 $\mu$ aluminium foil placed over this hole screened the crystal from direct light that may come from the X-ray generating elements.

Spectrometer and X-ray tube shared a common X-ray vacuum, which was maintained by a rotary molecular pump backed by a rotary oil pump. Exposures of the order of 300 to 500 milliamp.-minutes were used, the major part of which was made at 5 to 6 K.V. for the NiL lines. The remaining part of the exposure was of short duration at 17 to 18 K.V. for the reference lines. In general the NiL lines and reference lines appeared with one and the same setting of the kasette. Positive Kino film was used throughout.

The NiL lines were measured with respect to reference lines on either side. The reference lines used were cobalt $K\alpha_1$ and cobalt $K\alpha_2$ in the eighth order, and $NiK\alpha_1$ and $NiK\alpha_2$ in the ninth order. The dispersion was such that the CoK lines were separated by 1·14 mm., the NiK$\alpha$ lines by 1·31 mm., and the $CoK\alpha_1$ and $NiK\alpha_1$ by 22·43 mm. The $NiL\alpha_{1,2}$ line lay midway between the CoK$\alpha$ and NiK$\alpha$ lines. The $NiL\beta_1$ line lay close to the CoK$\alpha$ lines. The $NiL\alpha_{1,2}$ line was therefore measured directly by interpolation between the NiK and CoK lines in the following way :—The grazing angles for the NiK$\alpha$-doublet were calculated, and likewise the grazing angles for the CoK$\alpha$-doublet. The wavelengths adopted were those given in Siegbahn's ' Spektroskopie der Röntgenstrahlen ' (1931), p. 184, and the grating constants adopted were obtained from the same work, p. 48. For any point intermediate between the reference lines, simple proportionality was assumed between the distance of the point from one of the reference lines and the grazing-angular separation of the point from the same line. The proximity of the $NiL\beta_1$ line to the Co lines made it impossible to obtain both on the same film. The position of the $NiL\beta_1$ line relative to CoK and NiK was located by reference either to the NiK lines or to the chlorine $K\alpha_1$ and $K\alpha_2$ lines in the third order which (arising from impurities on the anticathode) appeared on all films.

In view of the breadth of the NiL lines, micro-photometric registrations were made of all films. All measurements were then made upon the photometer records of the films. For this purpose these records had super-

posed upon them the photometer record of a millimetre scale. The mechanical and optical principles of the micro-photometer have already been described *. Mechanical improvements in the instrument there described have recently been made.

To obtain the spectrum of pure nickel, a plate of metallic nickel was fitted to the anticathode. To obtain the nickel spectrum of the salts a plate of electrolytic copper was fitted to the anticathode, and the salts in powder form were pressed on to the copper plate, which was previously scratched. To provide against nickel impurities in the copper a long exposure was made with the copper only. No trace of nickel lines appeared. The only test as to whether chemical decomposition of the salts was taking place during an exposure was that of comparing a short exposure with a long exposure in the case of one of the salts ($Ni_2O_3$). From the two films it did not appear that decomposition was taking place.

The salts used in this investigation were $Ni_2O_3$, NiO, and NiS.

The following table gives in X.U. the wave-length of the position of maximum density of the $NiL\alpha_{1,2}$ and $NiL\beta_1$ lines, and the wave-length of the short wave-length edge of the lines, as they appeared in the photometer records. The precise position of the edge was estimated.

| | *Ni*. | $Ni_2O_3$. | *NiO*. | *NiS*. |
|---|---|---|---|---|
| $L\alpha_{1,2}$ max. | 14531±5 | 14529 | 14532 | 14531 |
| $L\alpha_{1,2}$ edge | 14500±4 | 14501 | 14500 | 14501 |
| $L\beta_1$ max. | 14244∓5 | 14246 | 14246 | 14246 |
| $L\beta_1$ edge | 14218±4 | 14216 | 14215 | 14215 |

The $NiL\alpha_{1,2}$ values for the pure metal are obtained from seven different films. ±5 in the case of the maximum and ±4 in the case of the edge represent the ranges in each case over which the wave-lengths obtained from the different films varied. The longer exposures seemed to shift the maximum towards the longer wave-

* Sandström, *Zeit. für Phys.* lxxxiv. p. 541 (1933).

lengths. For an unsymmetrical line, such as $NiL\alpha_{1,2}$ will be seen to be, this phenomenon can be explained in terms of the relation between photographic action and intensity.

The $NiL\beta_1$ values for the pure metal are obtained from three different films. There was less variation in the values obtained from different films ($\pm 1$). This was attributed to the smaller number of films taken, although in view of the approximate symmetry of the $\beta_1$ line (as will be seen), less variation might be expected in view of the facts just mentioned in connexion with the $L\alpha_{1,2}$ line. However, there seemed to be no good reason for not assigning the same ranges ($\pm 5$ and $\pm 4$) to $L\beta_1$. This was accordingly done.

The $NiL\beta_1$ line arises from an electronic transition from the $M_{IV}$ level to the $L_{II}$ level, and the $L\alpha_2$ line arises from an electronic transition from the $M_{IV}$ level to the $L_{III}$ level. From the frequency value of $NiL\beta_1$ and the difference in frequency values for $NiK\alpha_1$ and $NiK\alpha_2$, therefore, we may calculate the frequency, and so the wave-length, of $NiL\alpha_2$. The wave-length obtained in this way for the pure metal is 14535 X.U. This comes within the range of $NiL\alpha_{1,2}$ given in the table, and so provides no further information with respect to the $M_{IV}$ and $M_V$ levels, which differentiate $L\alpha_1$ and $L\alpha_2$. It is of interest to note, however, that the calculated value of $NiL\alpha_2$ is greater than the observed mean value of $NiL\alpha_{1,2}$, as would be expected theoretically.

In the case of each of the salts one exposure only was made, and the wave-lengths obtained lie within the range of values obtained for the corresponding quantity for the metal.

Fig. 2 shows a photometric record of the $NiL\alpha_{1,2}$ line from metallic nickel. The exposure was 300 milliamp.-minutes at 6 K.V. and 100 milliamp.-minutes at 16 K.V. The kasette was set for $NiL\alpha_{1,2}$ throughout. The $NiL\beta_1$ line just appears beneath the $CoK\alpha_1$.

Fig. 3 shows a photometric record of the $NiL\alpha_{1,2}$ and $NiL\beta_1$ lines from metallic nickel. The exposure was 650 milliamp.-minutes at 5 K.V. The kasette was set for $NiL\beta_1$ throughout.

These two figures show the characters of the metal L lines. In the case of $L\alpha_{1,2}$ the short wave-length edge is sharper than the long wave-length edge. $L\beta_1$

is rather symmetrical. The satellites on the short wave-length side of $L\alpha_{1,2}$ are apparent in the figures.

Fig. 2.

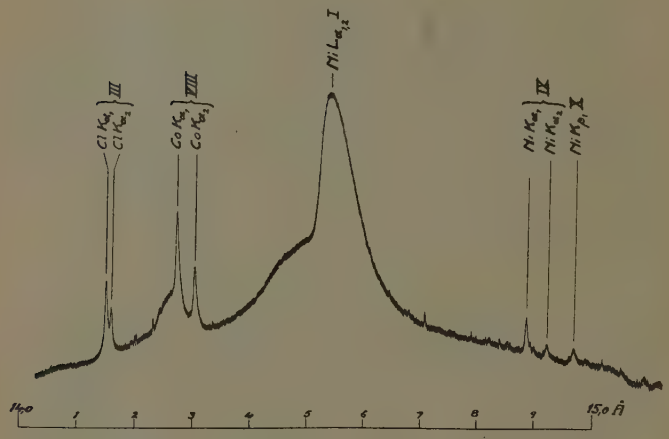
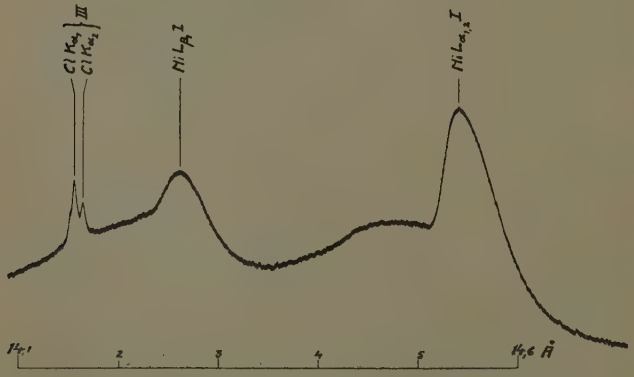



Fig. 3.




They extend to about 14402 X.U. and 14405 X.U. in figs. 2 and 3 respectively. The breadth of the L lines and the wave-length of all lines may be observed by references to the wave-length scale provided.

Fig. 4 shows the $NiL\alpha_{1,2}$ and $NiL\beta_1$ from $Ni_2O_3$. Satellites appear on the short wave-length edge of $L\beta_1$. The satellites associated with $L\alpha_{1,2}$ completely change in character. They are replaced by at least one sub-

Fig. 4.

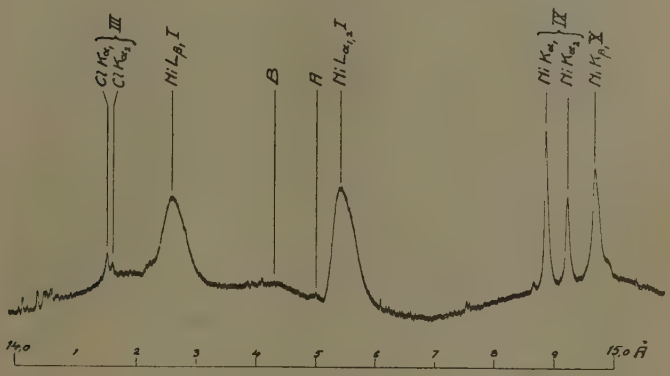
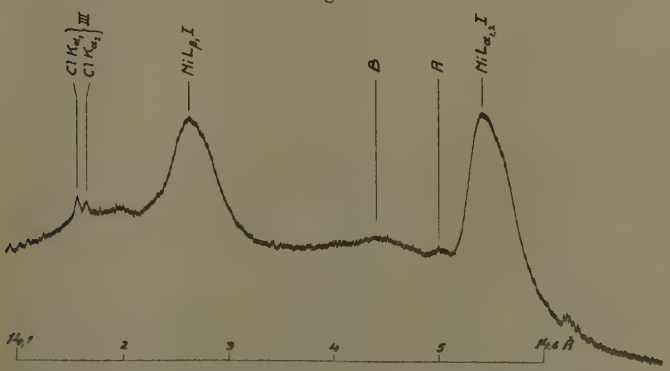



Fig. 5.




sidiary maximum, and possibly two subsidiary maxima, whose calculated wave-lengths are 14483 X.U. and 14418 X.U. The same approximately symmetrical character of $L\beta_1$ and definitely unsymmetrical character of $L\alpha_{1,2}$ may be observed.

Fig. 5 is another photometer record for the same salt. It is taken from an exposure of longer duration than

the previous. The general features are similar. There is a suggestion here and in fig. 4 of the doubling of the main Lα line as described by Thoraeus * for the trivalent nickel. The calculated wave-lengths of the two maxima

Fig. 6.

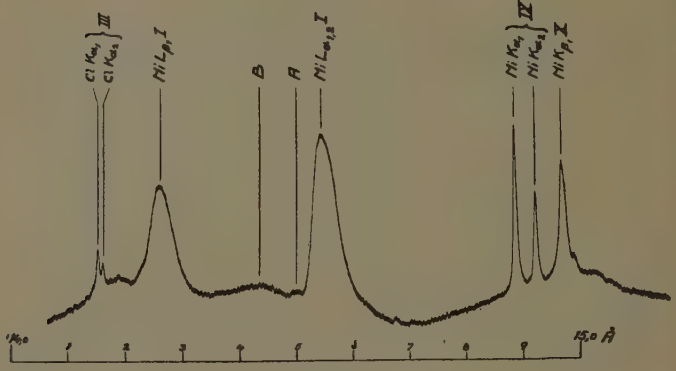
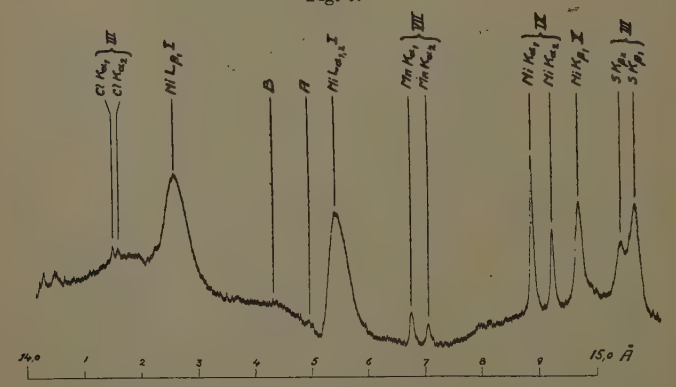



Fig. 7.




on the short wave-length side of $L\alpha_{1,2}$ are 14484 X.U. and 14421 X.U.

Fig. 6 shows the $NiL\alpha_{1,2}$ and Ni $L\beta_1$ from NiO. The same general features again appear. The calculated

* Thoraeus, *loc. cit.*

wave-lengths of the two maxima on the short wave-length side of $L\alpha_{1,2}$ are 14488 X.U. and 14431 X.U. The latter maximum is better defined here than in the two previous cases.

Fig. 7 shows the $NiL\alpha_{1,2}$ and the $NiL\beta_1$ lines from NiS. The same general characters again appear. The calculated wave-lengths of the two subsidiary maxima are 14483 X.U. and 14424 X.U.

The exposures of the films used for producing figs. 4, 6, and 7 were approximately the same as for the film recorded in fig. 2.

The following table gives the voltage differences between the $L\alpha_{1,2}$ maximum and the two subsidiary maxima, designated as "A" (long wave-length maximum) and "B" (short wave-length maximum). The voltage difference between $L\alpha_{1,2}$ and $L\beta_1$ for the metal is 17·2. The wave-lengths of the two subsidiary maxima are also tabulated.

| | Volts. | | X.U. | |
|---|---|---|---|---|
| | A−$L\alpha_{1,2}$. | B−$L\alpha_{1,2}$. | A. | B. |
| $Ni_2O_3$ ..... | 2·7 | 6·5 | 14483 | 14418 |
| NiO ..... | 2·6 | 5·9 | 14488 | 14431 |
| NiS ..... | 2·9 | 6·4 | 14483 | 14424 |

It gives me great pleasure to express my thanks to Prof. Siegbahn for the facilities granted, and for the advice and assistance offered, throughout the course of this work.

*Summary.*

The wave-lengths and characteristics of $NiL\alpha_{1,2}$ and $NiL\beta_1$ from metallic nickel, $Ni_2O_3$, NiO, and NiS are studied. The wave-length of the maximum (and of the short wave-length edge) is the same for all four substances to within about one in three hundred. Within the same accuracy the frequency difference between the L lines from the metal is equal to that between the $NiK\alpha_1$ and the $NiK\alpha_2$ lines. Features are described which characterize the spectra given by the salts.

Uppsala.
Feb. 18, 1935.

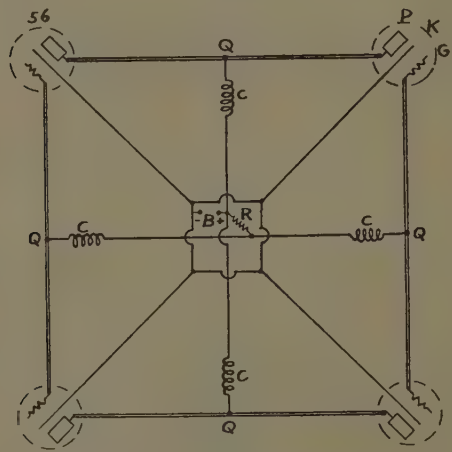
XXXV. *A Loop Transmitter.* *By* RONOLD KING, *A.B., M.S., Ph.D., Department of Physics, Lafayette College, Easton, Pennsylvania, U.S.A.**

[Plate III.]

## I. *Introduction.*

THE loop receiving antenna with its well-known directional characteristics is in common use. In the present paper the properties of the loop are incorporated in a new form of transmitting system in which

Fig. 2.




Circuit diagram of the laboratory transmitter.

C=choke coils. R=3000 ohm. resistance. (Instead of connecting the grid lead to +B through R, it may be joined directly to −B with R omitted.)

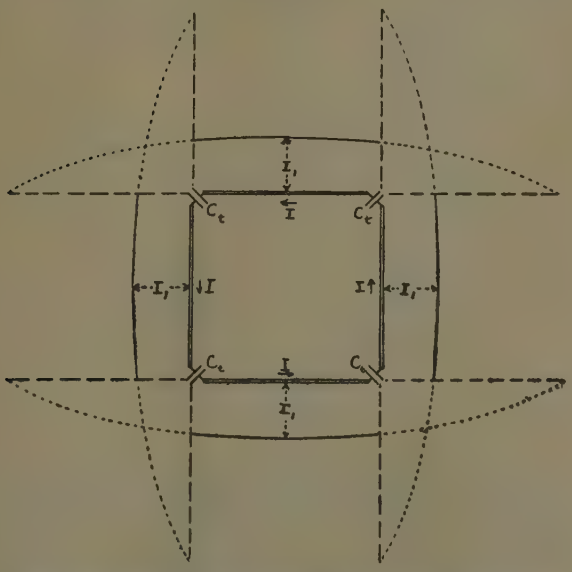
Note that heater connexions are not shown. The four heaters are connected in series; the leads are brought to the centre with the cathode wires.

a single loop serves both as a generator and as a directional radiator of electromagnetic waves. A special feature of one form of the device is the method of providing negative resistances in the loop circuit. As a basis

* Communicated by the Author.

for subsequent generalization, the mathematical theory will be discussed with special reference to a working model of the loop in the form of a square operating at a carrier frequency of $1{\cdot}02.10^8$ cycles per second. A photograph of this model is shown in fig. 1 (Pl. III.); its circuit diagram is drawn in fig. 2. The oscillating and radiating part of the circuit consists essentially of a square of brass tubing with a thermionic valve connected at each corner. Under proper operating conditions each triode introduces

Fig. 3.



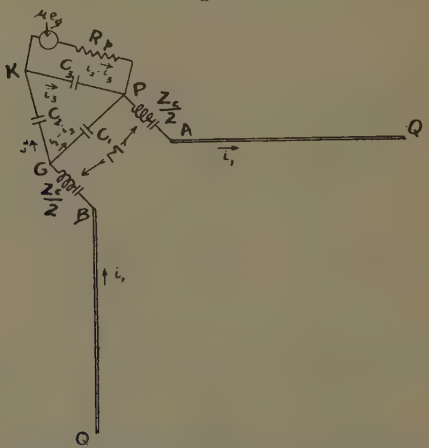

Schematic diagram of the current distribution on the square.
$C_t$ = input capacitance of each triode. $I = I_1 \cos \omega t$.

a negative resistance into the square, so that undamped oscillations are sustained. The frequency of these oscillations is determined by the dimensions of the square and by any other impedances (including, of course, those of the thermionic valves) which may be connected in series with it. The oscillating circuit may be considered to consist of four electrically similar segments, each of which

corresponds approximately to a regenerative oscillator of a type already analysed *.

The nearly sinusoidal current amplitude distribution along the conductors forming the square is shown schematically in fig. 3. If the dimension of the square is not too large a fraction of the generated wave-length, one may assume, to a first approximation, that the current amplitude is the same at all points along the square and in any impedances in series with it. This assumption is made throughout the following analysis, sinceit represents

Fig. 4.




Circuit diagram of one segment of the square, including the triode.

$Z_c$ = lumped impedance in series with each triode and the conductors of the square. M = total mutual inductance between PQ and GQ.

well the conditions actually encountered at ultra-short waves. Moreover, it leads to a simple, readily interpreted solution which is reasonably exact even when the current amplitude distribution is far from uniform.

## II. *The General Equation for a Square carrying a Uniform Sinusoidal Current.*

The mathematical problem which must be solved is to determine the fundamental vibration modulus of the circuit of fig. 4. This figure shows the complete circuit

* R. King, Proc. I. R. E xx. p. 1370 (August 1932).

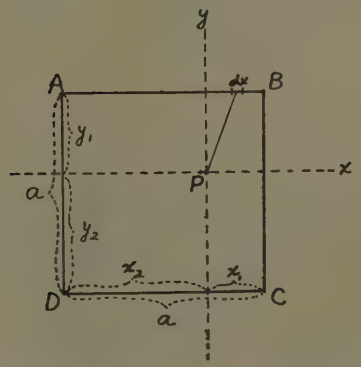
diagram of one of the four similar segments which together form the square.

The general equation for this circuit is obtained from one of the Maxwell field equations. In integral vector form this is:

$$\int(\mathbf{E}, d\mathbf{S}) = -j\omega \int \mathbf{B}_n d\sigma = \int -j\omega\phi. \quad . \quad . \quad . \quad (1)$$

In this equation **E** and **B** are the vector amplitudes, respectively, of the electric intensity measured in volts per cm., and of the magnetic flux density measured in webers per sq. cm.* Applied to the square this equation states that the line integral of the electric intensity

Fig. 5.




Construction for the determination of the flux density at any point in a square of dimension $a$.

around the perimeter is equal to the time rate of change of the total flux $\phi$ through the enclosed area. The definition of $\phi$ in terms of B is implied in (1).

In order to obtain the right side of (1) consider any point P in a square of dimension $a$. Choose the origin of coordinates at the point P as shown in fig. 5. Beginning with the formula for the flux density due to a dipole †, an expression for **B** may be derived in terms of the current

* 1 weber $= 10^8$ maxwells.

† Mason and Weaver, 'The Electromagnetic Field,' formula 247 (Univ. of Chicago Press, 1929); Abraham and Becker, 'The Classical Theory of Electricity and Magnetism,' formula 44$a$, p. 225 (Blackie and Son, 1932).

flowing in a straight conductor *. It is assumed that the current is sinusoidal in time and that its amplitude I is the same at all points around the square. If $y$ is the perpendicular distance from P to the conductor, and $r$ is the distance from P to the element $dx$, then the flux density amplitude B may be expressed as the integral of a series as follows ($k=\omega/c$):

$$B=yI/4\pi\int[(1/r^3+k^2/2r-k^4r/8+\ldots) -j(k^3/3-k^5r^2/30+\ldots)]\,dx. \quad (2)$$

For present purposes it will be sufficient to retain only the leading terms subject to the condition

$$r<<\lambda/2\pi. \quad (3)$$

Then the value of B at P due to the conductor AB is

$$B_{AB}=\frac{I}{4\pi}\left\{\left[\frac{x_1}{y_1\sqrt{x_1^2+y_1^2}}+\frac{y_1k^2}{2}\sinh^{-1}\frac{x_1}{y_1}\right] +\left[\frac{x_2}{y_1\sqrt{x_2^2+y_1^2}}+\frac{y_1k^2}{2}\sinh^{-1}\frac{x_2}{y_1}\right]-j\frac{k^3y_1a}{3}\right\}. \quad (4)$$

If the same current flows completely around the square, the contributions to B due to the currents in each of the four conductors will be perpendicular to the plane of the square, and all in the same direction The values of B due to the other three conductors are readily obtained from (4) by suitably interchanging the subscripts and $x$ and $y$. The total flux density due to the four conductors is, then, the algebraic sum of the four expressions so obtained.

The total flux $\phi$ through the square is calculated by integrating the flux density over the area. It is

$$\phi=\int_{d/2}^{a}\int_{d/2}^{a}B\,dx\,dy. \quad (5)$$

Here $d$ is the diameter of the conductors forming the square.

Upon carrying out the integration and neglecting certain small terms, the value of $\phi$ may be written as follows:

$$\phi=I(L-jR_r/\omega)\text{ webers/sq. cm.} \quad (6)$$

* R. King, "The Telegraphist's Equations at Ultra-High Frequencies," 'Physics,' vi. p. 121 (April 1935).

$$L = 8a(\log 4a/d - 1{\cdot}47 + 0{\cdot}22\,\omega^2a^2/c^2).10^{-9} \text{ henries}, \quad (7)$$

$$R_r = 20\omega^4a^4/c^4 = 20(2\pi a/\lambda)^4 \text{ ohms}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Here L is the inductance and $R_r$ the radiation resistance of the square.

The evaluation of the integral on the left side of (1) may be divided into two parts—the value of the integral around the square proper, and the value across the lumped impedances in series with the square.

By a generalization of Ohm's Law for conductors, one may write for the integral around the square the relation

$$\int(\mathbf{E}, \mathbf{ds}) = r\int(\mathbf{I}, \mathbf{ds}), \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where $r$ is the resistance per unit length of the conductors forming the square at the frequency $\omega/2\pi$. Since I and ds are both in the same direction, the integration around a square of dimension $a$ gives, simply,

$$\int(\mathbf{E}, \mathbf{ds}) = 4ar\mathrm{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

To evaluate the second part of the integral, let it be assumed that lumped impedances $Z_i$ and $Z_c$ are connected in series with each conductor in such a way that the circuit from the centre of one side to the centre of the next adjacent side is electrically identical with the other three corresponding circuits which form the square. For a lumped impedance Z one notes that

$$\int(\mathbf{E}, \mathbf{ds}) = -\int(dV/ds)\,ds = -\int dV = \mathrm{IZ}. \quad . \quad . \quad (11)$$

Since a term corresponding to (3) must be added for each lumped impedance, one has, on combining (11) with (10),

$$\int(\mathbf{E}, \mathbf{ds}) = 4\mathrm{I}(ra + Z_i + Z_c). \quad . \quad . \quad . \quad . \quad (12)$$

Here $Z_i = R_i + jX_i$ may be the input impedance of a thermionic valve.

Upon now combining (6) and (12) according to (1), and separating real and imaginary parts, one has two equations. They are:

(*a*) the condition for oscillation,

$$R_r + 4(ra + R_i + R_c) = 0; \quad . \quad . \quad . \quad . \quad (13)$$

(*b*) the frequency equation,

$$\omega L + 4(X_i + X_c) = 0. \quad . \quad . \quad . \quad . \quad (14)$$

The following notation is used :

$a$=dimension of the square of conductors.

$r$=resistance per unit length of the conductors of the square at the frequency $\omega/2\pi$. This is readily calculated from skin effect formulæ.

$R_i, X_i$=respectively, input resistance and reactance of each triode. To sustain oscillations $R_i$ must, of course, be negative.

$R_c, X_c$=respectively, resistance and reactance of any lumped impedance in series with the conductors of *each* thermionic valve.

$R_r = 20s^4$=radiation resistance of the entire square in ohms.

$L = 8a(\log 4a/d - 1{\cdot}47 + 0{\cdot}22s^2).10^{-9}$=inductance of the entire square in henries.

$s = \omega a/c = 2\pi a/\lambda$; $c$=velocity of light$=3.10^{10}$ cm./sec.

From equations (13) and (14) one may determine the operating characteristics of the transmitter.

An expression for the input impedance $Z_i$ of a thermionic valve is easily derived. Assuming a lumped mutual inductance M between the plate and grid leads of the circuit shown in fig. 4, one may write the following equations :

$$\left.\begin{aligned} &(i_1-i_2)/pC_1 = E_{pG}, \\ &i_2/pC_2 + R_p(i_2-i_3) - (i_1-i_2)/pC_1 + \mu e_g = 0, \\ &i_3/pC_3 - R_p(i_2-i_3) - \mu e_g = 0, \\ &e_g = -pMi_1. \end{aligned}\right\} \quad (15)$$

The impedance $Z_i$ is given by the equation,

$$Z_i = E_{PG}/i_1 = D(p)/A_{11}(p), \quad . \quad . \quad . \quad . \quad (16)$$

with

$$D(p) = \begin{vmatrix} 1/pC_1 & -1/pC_1 & 0 \\ -1/pC_1 - \mu pM & R_p + 1/pC_1 + 1/pC_2 & -R_p \\ \mu pM & -R_p & R_p + 1/pC_3 \end{vmatrix}.$$

$$. \quad . \quad . \quad (17)$$

With the familiar notation $Z_i = R_i + jX_i$, and upon recalling the relation

$$\mu = -C_2/C_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

one obtains the following expressions from (16):

$$X_i = -1/\omega C_i[1 + \mu^2 M/C_s R_p^2(\mu - 1) + 1/\omega^2 R_p^2 F C_s], \quad (19)$$

$$R_i = -1/R_p[\mu^2 M/C_i(\mu - 1) - (F - C_s)/\omega^2 C_i C_s F]. \quad . \quad (20)$$

These relations are subject to the condition

$$(1/\omega C_s)^2 << R_p^2. \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The following notation has been used:

$$\left.\begin{aligned} C_i &= (C_1C_2 + C_2C_3 + C_3C_1)/(C_2 + C_3), \\ C_s &= (C_1C_2 + C_2C_3 + C_3C_1)/(C_1 + C_2), \\ F &= C_2 + C_3. \end{aligned}\right\} \quad . \quad . \quad (22)$$

In most triode circuits the second and third terms in the brackets for $X_i$ are small compared with unity. In this case the input reactance reduces to the following very simple approximate form:

$$X_i = -1/\omega C_i. \quad . \quad . \quad . \quad . \quad . \quad (19a)$$

## III. *The Electromagnetic Field of a Square carrying a Uniform Current.*

The electric and magnetic field vectors **E** and **B** may be computed at any point in the wave-zone of the transmitting loop by superposing the fields set up by the individual conductors forming the square. Thus, if the origin of a system of rectangular coordinates (X, Y, Z) is located at the centre of the square, as shown in fig. 6, one proceeds as follows. From the expressions for the rectangular components of **E** in the wave-zone due to a dipole, it is easily shown that the following are the components of the amplitude of **E** due to an element of current of length $d$S parallel to the Y-axis:

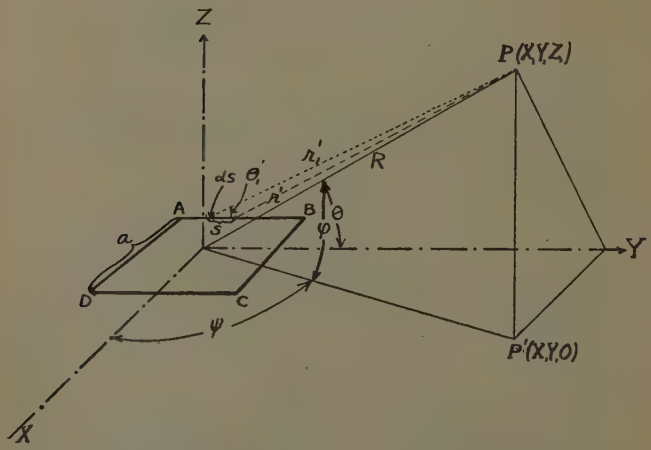
$$\left.\begin{aligned} dE_x &= j30I\frac{\omega}{c}\frac{x_1' y_1'}{r_1'^3} e^{-j\frac{\omega}{c}r_1'} dS, \\ dE_z &= j30I\frac{\omega}{c}\frac{z_1' y_1'}{r_1'^3} e^{-j\frac{\omega}{c}r_1'} dS, \\ dE_y &= j30I\frac{\omega}{c}\left(\frac{y_1'^2 - r_1'^2}{r_1'^3}\right) e^{-j\frac{\omega}{c}r_1'} dS. \end{aligned}\right\} \quad . \quad . \quad (23)$$

Upon integrating these expressions over the length $a$ of the conductor and neglecting certain small terms, one obtains the components of the electric intensity due to the conductor AB forming one side of the square:

$$\left.\begin{aligned} E_x &= (j60IX/R^2)e^{-j\omega r'/c}\sin(\omega aY/2cR), \\ E_z &= (j60IZ/R^2)e^{-j\omega r'/c}\sin(\omega aY/2cR), \\ E_y &= -j60I(1/Y - Y/R^2)e^{-j\omega r'/C}\sin(\omega aY/2cR), \end{aligned}\right\} \quad (24)$$

$$r' = \sqrt{(X+a/2)^2+Y^2+Z^2} \doteq R + aX/2R. \quad (25)$$

Fig. 6.




Representation of the square in a system of coordinates.

The rectangular coordinates of the point P with reference to the centre of the square are X, Y. Z. The polar coordinates are the radius vector R, the angle $\phi$ between this radius and the XY-plane (which is the plane of the square), and the angle $\psi$ between the projection of R on the XY-plane and the X-axis.

Three sets of corresponding expressions are obtained for the components of E due to the other three conductors. Upon combining the four terms contributing to each component of **E**, and noting that

$$\left.\begin{aligned} e^{-j\omega r'/c} - e^{-j\omega r''/c} &= -2j\sin(\omega aX/2cR)e^{-j\omega R/c} \\ e^{-j\omega r'''/c} - e^{-j\omega r''''/c} &= -2j\sin(\omega aY/2cR)e^{-j\omega R/c} \end{aligned}\right\}, \quad (26)$$

where, corresponding to (25),

$$\left.\begin{aligned} r'' &\doteq \mathrm{R}-a\mathrm{X}/2\mathrm{R} \\ r''' &\doteq \mathrm{R}+a\mathrm{Y}/2\mathrm{R} \\ r'''' &\doteq \mathrm{R}-a\mathrm{Y}/2\mathrm{R} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (27)$$

it is easily shown that

$$\left.\begin{aligned} \mathrm{E}_x &= 120\mathrm{I}e^{-j\omega\mathrm{R}/c}(1/\mathrm{X}) \sin(\omega a\mathrm{X}/2c\mathrm{R}) \sin(\omega a\mathrm{Y}/2c\mathrm{R}) \\ \mathrm{E}_y &= 120\mathrm{I}e^{-j\omega\mathrm{R}/c}(1/\mathrm{Y}) \sin(\omega a\mathrm{Y}/2c\mathrm{R}) \sin(\omega a\mathrm{X}/2c\mathrm{R}) \\ \mathrm{E}_z &= 0 \end{aligned}\right\}. \quad (28)$$

From fig. 6 it is clear that one may write

$$\mathrm{X}=\mathrm{R}\cos\phi\cos\psi\ ;\ \mathrm{Y}=\mathrm{R}\cos\phi\sin\psi.$$

The amplitude of E is then given by

$$\mathrm{E}=120\mathrm{I}\left\{\frac{\sin(\omega a/2c\cos\phi\cos\psi)\sin(\omega a/2c\cos\phi\sin\psi)}{\mathrm{R}\cos\phi\cos\psi\sin\psi}\right\}. \quad (29)$$

The direction of E is that of $\psi$, *i.e.*, E is perpendicular to the radius vector drawn from P to the centre of the square, and parallel to its plane.

By expanding (29) in series it can be shown that the trigonometric factor of 120I/R reduces to $\mathrm{A}\dfrac{\sin\mathrm{A}}{\cos\phi}$ for $\mathrm{A}<\pi/2$, and to $\mathrm{A}^2/\cos\phi$ for $\mathrm{A}\leqq\pi/6$. Here $\mathrm{A}=(\omega a/2c)\cos\phi$. In the latter case (29) assumes the very simple form

$$\mathrm{E}=(30\mathrm{I}s^2\cos\phi)/\mathrm{R} \text{ with } s=\omega a/c=2\pi a/\lambda. \quad . \quad (30)$$

The complete expression for the electric field due to a square carrying a current of amplitude I is, then,

$$\mathrm{E}=30\mathrm{I}(s^2/\mathrm{R})\cos\phi\cos\omega(t-\mathrm{R}/c) \text{ volts/cm.} \quad . \quad (31)$$

In the wave zone the magnetic flux density vector B is always perpendicular to E and to the radius vector drawn from P to the centre of the square. It is given by

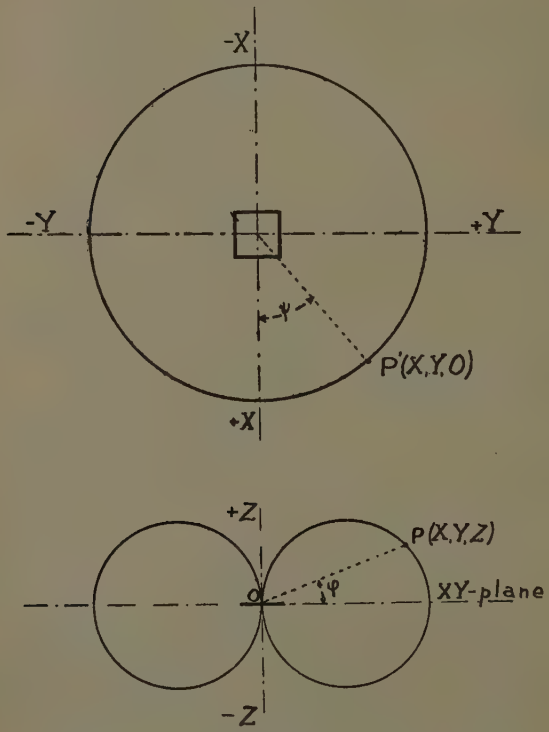
$$\mathrm{B}=(\mathrm{I}s^2/4\pi\mathrm{R})\cos\phi\cos\omega(t-\mathrm{R}/c) \text{ webers/sq. cm.} \quad (32)$$

The electromagnetic field of the square is thus determined.

In the derivation of (31) and (32) it is assumed that the dimension of the square is sufficiently small compared with the wave-length that the relation, $\sin(s)\doteq s$, is a good approximation. Since this assumption is essentially

the same as that under which the current amplitude distribution along the conductors is approximately uniform, no new restriction is introduced. If $a \leq \lambda/2\pi$, the above formulæ are excellent representations of the magnitudes of the electromagnetic field vectors.

Fig. 7.




Amplitude of the electromagnetic field in the wave-zone.

Upper figure : The radius OP is proportional to the amplitude of **E** (or **B**), as measured from the X-axis by the angle $\psi$, in the plane of the loop.

Lower figure : The length OP is proportional to the amplitude of **E** (or **B**), in a direction making the angle $\phi$ with the plane of the loop.

The field defined by (31) and (32) is diagrammatically shown in fig. 7. It is the same as that due to a magnetic

dipole perpendicular to the plane of the square, and is placed at its centre. This is true, of course, only in the wave-zone. The loop transmitter might, therefore, be called a magnetic dipole. Since the field due to a magnetic dipole is the same as that due to the familiar electric dipole, but with electric and magnetic vectors interchanged, it follows that little more need be said to describe the field characteristics of the loop transmitter, or of combinations of such loops. In so far as the wave-zone is concerned, the electromagnetic field of a single loop, or of systems of sufficiently well separated loops, may be derived from that of equivalent dipoles.

In the preceding section the radiation resistance of the square was derived in conjunction with the condition for oscillation of the loop. It may also be derived in the more usual manner from Poynting's theorem. Thus the total power radiated is

$$P=\int[\mathbf{E},\mathbf{B}]\,d\mathbf{V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

This integral gives

$$P=20s^4I^2\cos^2\omega(t-R/c). \quad . \quad . \quad . \quad . \quad (34)$$

Since the mean value of $I^2\cos^2\omega(t-R/c)$ is $I_{\text{eff.}}^2$, it follows that

$$P=R_r I_{\text{eff.}}^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

where

$$R_r=20s^4=20(2\pi a/\lambda)^4. \quad . \quad . \quad . \quad . \quad (36)$$

This value agrees with that previously obtained in (8).

## IV. *The Generalized Loop Transmitter.*

In the preceding sections the frequency and field characteristics of a loop transmitter in the form of a square were discussed. In the present section possible modifications in the construction and operation will be considered briefly.

The electric and magnetic fields of a loop carrying a uniform current are, to a first approximation, independent of the shape of the loop, providing this preserves general symmetry with respect to the centre. That is, the transmitting loop may be in the form of a circle, a square, a hexagon, etc., and the electromagnetic field in the wave zone will still approximate very closely that described

for the square. If the shape is made rectangular, rhombic, or elliptical, the electric field pattern will not be centrally symmetrical.

It follows from the frequency equation (14) that the frequency generated by the loop depends as well upon the dimensions of the loop itself, as upon the reactive components of any impedances which may be connected in series with it. The frequency generated and the size of the loop may, therefore, be varied independently over quite a wide range if suitable reactances are used in series with the loop. Such reactances must, of course, be connected in a way preserving the electrical similarity of the segments of the loop. It is to be remembered that the radiation resistance of the loop decreases rapidly as its dimensions are reduced, while the wave-length is maintained the same. In the case of the square, for example, $R_r$ is proportional to $(a/\lambda)^4$ for $a \leqq \lambda/2\pi$.

One of the special features of the loop transmitter as described above is the new method of supplying power by means of negative resistance devices suitably spaced around the loop. Such negative resistances may be triodes, or they may be any other convenient form of power converting device. If triodes are used plate circuit modulation is convenient. It is, of course, perfectly possible to supply power in the conventional way by coupling an oscillating-amplifier inductively to one or more of the radiating elements of the loop.

## V. *Applications and Special Properties of the Loop Transmitter.*

Applications of the loop type transmitter would depend primarily upon its directional properties combined with its constructional characteristics. With reference to the former, little need be said other than to refer to the field patterns of fig. 7. From these it is clear that the loop may be used to radiate equally in all directions along a horizontal plane by placing it parallel to this plane. If, on the other hand, the loop is placed in a vertical position, the radiated energy is a maximum in two opposite directions along the horizontal, and a minimum in the two directions at right angles to these. In this way the loop may be adapted to either directional or non-directional transmission.

Constructionally the loop differs from the ordinary half-wave antenna, the field patterns of which it resembles, in having its greatest physical dimension in the plane of maximum radiation, whereas the half-wave antenna is longest in the direction of minimum radiation. In some cases this difference in the relative directions of maximum radiation and maximum physical length might be of great advantage.

The loop transmitter seems especially well adapted to short waves. A portable short wave transmitter, for example, may be constructed easily. This could be very compact, powerful, and strongly directional. The fact that oscillators, radiators, and reflectors are combined in a single, simple unit, the physical dimensions of which may be kept as small as desired, would seem to be especially important. The available power output would depend upon the number of similar segments arranged around the loop, and upon the type of valve used. With power oscillators very great output should be possible.

No attempt will be made to outline possible commercial or technical uses of the loop transmitter. It would seem that it might have advantages for use on airplanes, for airplane beacons, for various kinds of short-wave transmission, or even for commercial broadcasting. The purpose of the present paper has been to describe the transmitter and to derive its fundamental mathematical-physical characteristics.

## VI. *Working Models.*

The general theory outlined above has been completely verified by applying it to two laboratory models. The first is that shown in fig. 1 (Pl. III.) the second is a very much smaller one operating at a wave-length of 176 cm. and using the new "acorn" RCA 955 valves. The frequency equation was carefully verified, and the computed value of wave-length was found to agree well with the measured one. The field pattern was found to agree with the results of the theory shown in fig. 7. A study of the frequency-plate voltage characteristics indicated that in the case of the transmitter shown in fig. 1 the generated frequency is practically independent of the plate voltage. This means that no appreciable frequency modulation

would take place if plate circuit amplitude modulation were used.

Much more could be added with reference to laboratory measurements. Since this would be largely a repetition in a special case of the theory, further discussion will be omitted.

The writer is indebted to Dr. C. McC. Gordon, Professor of Physics at Lafayette College, for reading the manuscript and suggesting numerous improvements.

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## XXXVI. *The Roots of* $\cos z = z$.

*By* Ludwik Silberstein, *Ph.D.**

So far as I know, the roots of the simple transcendental equation

$$\cos z = z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

have not been investigated, probably because this equation, unlike the extensively treated one, $\tan x = x$, does not, or at least did not hitherto, occur in connexion with physical problems.

In spite of this practical inapplicability of (1), it has seemed interesting and worth while to investigate its roots.

As the reader will see at once from a graph of $u = \cos z$ and $u = z$, our equation has (unlike $\tan z = z$) *only one real root*, represented by the unique intersection point of the simple wave-line and the straight line bisecting the right angle between the axes of the diagram. The approximate value of this root is readily found either by Newton's method or (if $x_1$ be a "first" approximation *smaller* than the actual root) by the obvious successive approximations

$$x_2 = \cos x_1, \quad x_3 = \cos x_2, \text{ etc.},$$

which can be written compactly

$$x = \ldots\ldots \cos\cos\cos x_1.$$

To five decimal places the real root is

$$x = 0{\cdot}73909 = 42°{\cdot}346. \quad . \quad . \quad . \quad . \quad (2)$$

* Communicated by the Author.

But since the contemplated equation is a transcendental one or, if the reader prefers, an algebraic equation of an infinite degree,

$$1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots . - z = 0,$$

it manifestly must have an infinity of roots, all of which, except (2), will be complex.

In order to find them, put

$$z = x + iy,$$

where $i = \sqrt{-1}$ and $x, y$ are real. Then the equation (1) will split into the two real equations

$$\cos x . \cosh y = x, \quad \sin x \sinh y = -y. \quad . \quad . \quad (1a)$$

The required roots, points of the $z$-plane which may be called *the nodes* of the function

$$f(z) = \cos z - z, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

are the crosses of the two curves

$$\cosh y = x/\cos x, \quad . \quad . \quad . \quad . \quad . \quad (C_1)$$

$$\sin x = -y/\sinh y, \quad . \quad . \quad . \quad . \quad . \quad (C_2)$$

with $x$, $y$ as, say, the orthogonal co-ordinates of a point of the $z$-plane. These nodes can be determined by successive approximations, the process being graphically represented by crinkly paths tending to, or oscillating in decreasing amplitudes about, one or other node. In order to pick out the proper starting-points of these "paths" (*i. e.*, the first approximations) and the right sense of describing them, the two curves must be drawn, or at least sketched roughly, through a few calculated points of $C_1$ and $C_2$.

For $x \doteq 0{\cdot}739$, *i. e.*, for $x/\cos x = 1$, we have, by $(C_1)$, $\cosh y = 1$, $y = 0$, so that both curves pass through the node on the real axis, as they should.

Either curve consists of an infinity of disjointed branches.

One branch of $C_2$ is obviously the whole $x$-axis itself. There is no other branch of $C_2$ in the interval

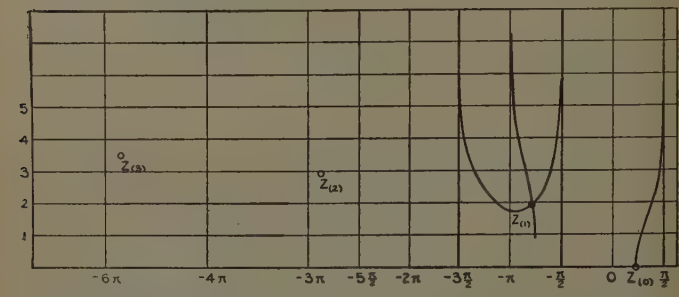
$$-\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}.$$

A few points of the branch of $C_1$ within this interval are

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x=$ | 0·75 | ·80 | ·90 | 1·00 | 1·20 | 1·40 | 1·50 | 1·55 | $\pi/2$ |
| $y=$ | ·23 | ·54 | ·92 | 1·23 | 1·87 | 2·80 | 3·74 | 5·0 | $\infty$ |

These are sufficient for drawing this branch (see figure). Manifestly there are in this interval no nodes other than that, (2), on the $x$-axis.

The next branch of $C_1$ is contained between $x=-\pi/2$ and $-3\pi/2$, with a similar one between $\pi/2$ and $3\pi/2$. While there are no $C_1$-points between $-3\pi/2$ and $-5\pi/2$, there is a branch of $C_1$ between $-5\pi/2$ and $-7\pi/2$, similar in character to the preceding one, and a $C_1$-branch to match within the interval $5\pi/2$ to $7\pi/2$, and so on, as shown in the figure.




The graph shows that the $-\pi/2$ to $-3/\pi 2$ branches of $C_1$ and $C_2$ intersect at about

$$x=-2\cdot5, \quad y=1\cdot8.$$

In fact, for $x=-2\cdot5$, equation $(C_1)$ gives $y=1\cdot80$, and to this value corresponds, by $(C_2)$, $x=-2\cdot52_4$. Having thus roughly located the complex root

$$z_1=x_1+iy_1=-2\cdot5+1\cdot8i,$$

let us determine it more accurately by a successive application of the two equations $(C_1)$, $(C_2)$, to wit:—

Starting with $y_1=1\cdot82$, say, let us compute $x_1$ from $\sin x=-y/\sinh y$, then $y_2$ from $\cosh y=x/\cos x$; next, $x_2$ from the first of these two equations, and so on. In this way I find

$y_1=1\cdot82$ $\quad y_2=1\cdot808$ $\quad y_3=1\cdot8096$ $\quad y_4=1\cdot8093$

$x_2=-2\cdot491$ $\quad x_2=-2\cdot4863$ $\quad x_3=-2\cdot4870$ $\quad x_4=-2\cdot4869.$

This, then, is a very rapidly converging process (as could be expected from the graph), and it gives, to four decimal places, the required root, with its conjugate (which obviously is also a root of $\cos z = z$),

$$z_{(1)} = -2{\cdot}4869 \pm 1{\cdot}8093i.$$

This is a pair of roots nearest to the real one

$$z_{(0)} = 0{\cdot}73909.$$

To find the roots in the zone $-3\pi \leqq \times \leqq -5\pi/2$, the same process was used, but, to avoid starting with a too low $y$, I have taken $y_1 = 2{\cdot}5$. This led to

$$x_1 = -9{\cdot}00 \quad x_2 = -9{\cdot}12 \quad x_3 = -9{\cdot}112 \quad x_4 = -9{\cdot}1110.$$
$$y_2 = 2{\cdot}981 \quad y_3 = 2{\cdot}949 \quad y_4 = 2{\cdot}9505.$$

The process is again a rapidly converging one, and gives, to four decimal places, the pair of roots

$$z_{(2)} = -9{\cdot}1100 \pm 2{\cdot}9505i.$$

Similarly I find

$$z_{(3)} = -15{\cdot}488 \pm 3{\cdot}4566i.$$

The evaluation of further roots may be left to the reader.

For large indices $n$ in $z_{(n)}$ the value of $y$ is large, and the two equations become, approximately,

$$\sin x = = 2ye^{-y}, \quad \cos x = 2xe^{-y},$$

whence

$$x \tan x = -y, \quad x^2 + y^2 = \tfrac{1}{4}e^{2y},$$

and the latter equation may as well be written, asymptotically,

$$x^2 = e^{2y}/4.$$

To judge from these few complex roots, showing the spacings $\frac{1}{2}\Delta_{12}x = 3{\cdot}315$, $\frac{1}{2}\Delta_{23}x = 3{\cdot}189$ and $2\Delta_{12}y = 2{\cdot}2824$, $2\Delta_{23}y = 3{\cdot}0122$, one feels inclined to expect that the $x$-intervals of the roots will tend (decreasingly) to $2\pi$ and their $y$-intervals (increasingly) to $\pi/2$. Wherever we turn (in Analysis, not in Theory of Numbers), sooner or later a monotonous distribution of characteristic points, in this case " nodes " or roots, sets in.

Rochester, N.Y.
March 30, 1935.

XXXVII. *The Balancing of Valve Circuits.* *By* JOHN J. DOWLING, *M.A.*, *F.Inst.P.*, *and* CORMAC O'CEALLAIGH, *M.Sc.*, *University College, Dublin* *.

SEVERAL workers † have investigated the possibility of so balancing the output current from a thermionic valve that a sensitive galvanometer may be used to indicate output variations without, at the same time, responding to fluctuations arising in the valve or batteries themselves. Previous workers are mainly interested in "electrometer" valve circuits; we, however, had in view the balancing of circuits of the "ultramicrometer" type ‡, but the methods here to be described are equally efficacious in all kinds of valve circuits.

As Du Bridge (*loc. cit.*) indicates there are two distinct effects:—Drifting and irregular fluctuations of the galvanometer reading.

Drifting is mainly due to slow changes in battery voltages; but it may also be caused by slow progressive changes in the valve. Fluctuations are principally associated with irregular emission from the cathode, but sometimes arise from a defective battery. Thus, the problem is of a dual nature, and two distinct devices must necessarily be employed for complete compensation. One type of drift, due to thermal causes, persists for some time after lighting up the valve. This cannot be eliminated by any method, but it ceases after an hour or two. Wynn-Williams employs a bridge arrangement with two valves. This cannot, of course, deal with fluctuations; indeed, the use of two valves doubles the number thereof. On the other hand, Du Bridge avails of the space charge grid current in his electrometer tetrode circuit (fig. 1), and, to a very remarkable extent, he was able to obtain compensation for irregular emission effects. Unfortunately, a fundamental assumption of Du Bridge is not quite generally valid.

In effect this is that when emission changes equal

* Communicated by the Authors.

† Wynn Williams, Proc. Camb. Phil. Soc. xxiii. p. 810 (1927); Soller, Rev. Sc. Inst. 3, viii. p. 416 (1932); Du Bridge, *ibid.* 4, x. p. 532 (1933); giving full list of references; Turner, *ibid.* 4, xii. p. 665 (1933); Donzelet, Pierret, Divoux, *Journal de Physique* 169 S, Dec. 1933.

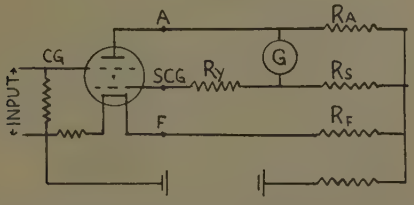
‡ Dowling, Phil. Mag. xlvi. (July 1923), and other publications.

percentage changes occur in both anode current and space charge grid current.

This " Du Bridge's Condition " can, however, be made to hold by connecting resistances externally to the valve circuits (anode and grid). This leads to one successful solution of the problem and is described fully below (" matched characteristic balance "). In the other of our methods emission effects and H.T. battery changes are separately compensated by independent adjustments of suitable external circuits.

The present work differs in a very outstanding way from that of Du Bridge and several others. In our circuits (" ultramicrometer circuits ") we employ ordinary valves, usually triodes, running under normal conditions. The anode and grid currents are of several milliamperes, and the H.T. battery may be upwards of 100 volts.

Fig. 1.




Their electrometer valves have outputs of less than half a milliampere, and are run from low voltage batteries of large capacity. The conditions we deal with impose much more severe tests on the effectiveness of the balance. However, although most of our work was concerned with these oscillatory circuits, the conclusions we arrive at are applicable to almost any valve circuit using space charge tetrodes and to many with triodes or pentodes.

In the following discussion the term anode is primarily used in its usual significance, but the reader must remember that all the statements which follow refer equally to a space charge grid, or other electrode, which is at a positive potential.

The linear part of the anode-current voltage valve diagram (fig. 2) may be represented by the equation

$$I = \frac{E - T}{\rho + R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.)$$

The tangential intercept OT may be regarded as a virtual counter-voltage T in the circuit.

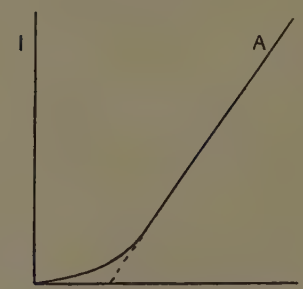
Now Du Bridge's assumption amounts to saying that *

$$\frac{\frac{\partial I_A}{\partial e}}{I_A} \text{ and } \frac{\frac{\partial I_S}{\partial e}}{I_S}$$

are independent of the resistance of the circuits external to the valve. However, from equation (I.) we have

$$\frac{\frac{\partial I}{\partial e}}{I} = \frac{\frac{\partial T}{\partial e}}{E-T} - \frac{\frac{\partial \rho}{\partial e}}{\rho+R} \quad . \quad . \quad . \quad . \quad . \quad (II.)$$

Fig. 2.




The second dextral term obviously depends on R, since $\rho$ varies with the cathode emission.

Complete balancing of the galvanometer can be obtained provided that three sets of conditions are fulfilled, viz. :—

A. The anode and grid currents must be balanced in some form of bridge.

B. This condition of balance must be independent of external battery voltage variations.

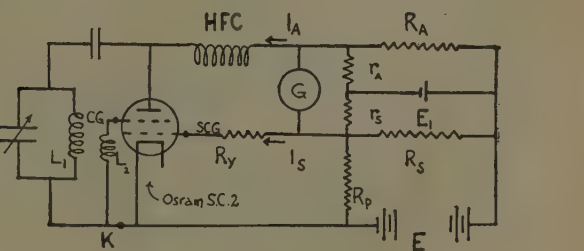
C. It must also be independent of cathode emission changes.

* $I_A$=anode current; $I_S$=space charge grid current; $e$=rate of cathode electron emission; E=applied voltage; $\rho$="differential" resistance of valve; R=external series resistance.

Now condition A can be readily obtained for any single battery voltage; but, from equation (I.) it is clear that the balance between anode and grid currents will be disturbed if E changes; for T may be different for the two circuits. Such is usually the case. An independent adjustment is then necessary to achieve condition B. These being both satisfied, condition C has to be dealt with. The "zero-balance" method, now to be described, was developed on these lines.

As a preliminary to this work a series of tests satisfied us that the valves we purposed using had characteristics of the type of fig. 2. When used in oscillatory ("ultramicrometer") circuits, this (*direct*) current-voltage characteristic changed considerably, but still conformed

Fig. 3.




to the desired shape. It is to be understood that the "ultramicrometer" circuit is such that, when the condenser C varies in capacity, the (d.c.) anode current varies a corresponding amount *.

*Zero Balance Circuit.*

Figs. 3 and 4 show two of our ultramicrometer circuits ("zero balance" type). The corresponding parts of figs. 3 and 4 are lettered similarly, but we shall specifically refer to fig. 3 in the following discussion.

(It is to be noticed that the ultramicrometer (H.F.) circuit on the left hand can be omitted and the control grid C.G. could be connected, much as in Du Bridge's arrangement (fig. 1), to some other circuit.)

In both figures E is a small cell, preferably one of

* Dowling, Phil. Mag. *loc. cit.*

small temperature coefficient. For no deflexion of the galvanometer the adjustment is

$$R_A I_A = R_S I_S \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (III.)$$

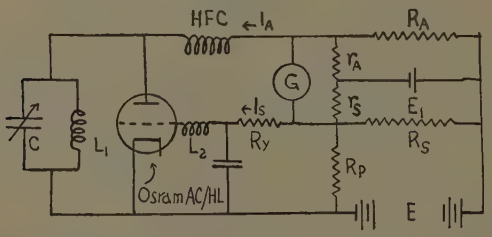
If, in addition,

$$E_1 = R_A I_A = R_S I_S \quad , \quad . \quad . \quad . \quad . \quad . \quad (IV.)$$

the connexion at P may be made without disturbing the galvanometer. Now, suppose the filament temperature is considerably lowered, so lessening the emission $e$; $I_A$ and $I_S$ both diminish, becoming $I_A - a$ and $I_S - b$. In general condition (III.) is no longer satisfied, much less (IV.).

We can, however, alter the ratio $r_A$ to $r_S$ so that the galvanometer returns to zero; and it will now be found that large variations of the filament emission are possible

Fig. 4.




*without deflecting the galvanometer.* The theory of this adjustment is simple.

For " current balance " as before

$$R_A I_A = E_1 = R_S I_S \quad . \quad . \quad . \quad . \quad . \quad (IV.)$$

when the emission changes this ceases to hold, for generally

$$\frac{a}{I_A} \neq \frac{b}{I_S};$$

as described $r_A$ and $r_S$ are adjusted to regain the balance, leaving $R_A$ and $R_S$ unaltered. Applying Kirchhoff's laws, we readily arrive at the condition

$$\frac{a}{\frac{1}{R_A} + \frac{1}{r_A}} = \frac{b}{\frac{1}{R_S} + \frac{1}{r_S}} \quad . \quad . \quad . \quad . \quad . \quad (V.)$$

This can obviously be satisfied simultaneously with (IV.), provided that the values of $R_A$ and $R_S$ are sufficiently large *. Therefore, for such emission changes as are possible without altering the *ratio* of $a/b$, the system is balanced in respect of emission.

The function of the resistance $R_y$ is important. It is chosen of such a magnitude that while variations of emission (or H.T. voltage) cause $I_S$ to change in the same sense as $I_A$; on the contrary, variations of the ultramicrometer condenser make $I_A$ and $I_S$ change in opposite senses. Thus a "push-pull" effect is obtained, which enhances the sensitivity of our ultramicrometer device, while the possibility of balancing the circuit is not affected.

There remains the compensation for HT battery voltage changes. The potential balance just described could have been utilized for this purpose, but, it being no longer available, recourse is made to another device †. The fact that one or other of the two currents $I_A$ and $I_S$ will vary more rapidly with H.T. voltage is the cause of H.T. battery fluctuation and drift effects. The resistance $R_p$ is connected as a shunt to one of the valve paths (grid in fig. 3). This may be adjusted, quite easily, by a process of trial and error until changes in the H.T. battery of a few volts produce no effect on the galvanometer. Simultaneous readjustment of $R_S$ (and possibly $r_S$) is necessary. The whole process takes but a few minutes ‡.

Provided that the valve is functioning well up on the linear range (fig. 2), these adjustments are so good that a deflexion of $10^{-8}$ ampere is not registered even with a 5 per cent. change either of anode battery voltage or filament heating current, whether gradually or suddenly applied. The balance condition yielded by this circuit is excellent, but for the sake of brevity we shall omit further practical details and the results obtained; for

* It is now clear that one of the resistances $r_A$ or $r_S$ can be omitted; this is advantageous, as it removes the shunting effect of $r_A+r_S$ from the galvanometer.

† This step was first made in this connexion, but on consideration it led to the development of the "matched characteristic balance" (*q. v. infra*).

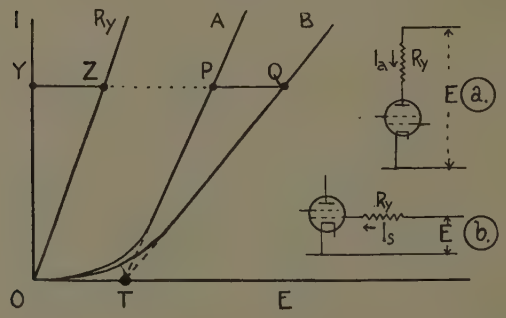
‡ It is *very important* that the applied voltage be changed in such a way that it *never drops below* say 90 per cent. of working voltage. This may be done by switching in and out a series resistance.

our second type of circuit, now to be described, is equally *effective and much simpler.*

*The "Matched Characteristics" Balanced Circuit.*

The basic idea suggested itself in considering the function of the resistance $R_p$ in the previous case. It is that the current voltage characteristic of a valve path (anode-filament or grid-filament path) can be effectively altered by coupling resistances either in series or in parallel therewith. The principle is readily understood by a consideration of figs. 5 and 6.

Fig. 5.



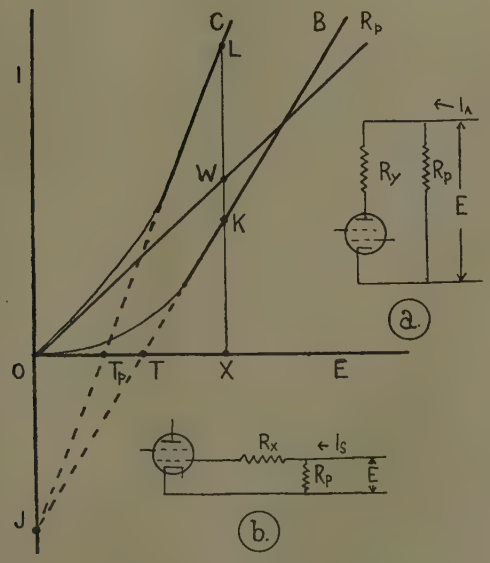

The typical valve characteristic (fig. 2) is represented by OPA in fig. 5. The line OZ represents the variation of current with P.D. in the resistance $R_y$ alone. When a current equal to OY is passing through the series combination (fig. 5, *a* or *b*) the total drop of potential E is represented by YZ+YP or YQ if PQ=YZ. Thus *characteristic of the combination* is OQB of which, it is easy to see, the linear part, when produced, meets the voltage axis in the same point T as does the original valve characteristic. In other words, adding a resistance $R_y$ in series with the valve path changes $\rho$ to $\rho_s=(\rho+R_y)$ without altering the counter voltage T.

It is different if the resistance $R_p$ is placed in parallel. In fig. 6, OW represents the variation of current with P.D. through $R_p$ (fig. 6, *a* or *b*). In this case this P.D. is E, and we have to add the current XW carried

by $R_p$ to that carried by the valve, namely, XK. Thus we make KL=XW and obtain the characteristic OLC for the combination. In this case the linear parts of both characteristics radiate from a common point J on the current axis. The counter voltage $T_p$ of the parallel combination is less than that of the valve T. The resistance becomes, of course, smaller, being given by

$$\frac{1}{\rho_p}=\frac{1}{\rho_s}+\frac{1}{R_p}=\frac{1}{\rho+R_y}+\frac{1}{R_p}.$$

Fig. 6.




Thus $\rho_p$ may be made either greater or less than $\rho$ for the valve alone.

The characteristic of one valve path being

$$I_1=\frac{E-T_1}{\rho_1}. \quad . \quad . \quad . \quad . \quad . \quad (1.A)$$

It should thus be possible to match it by reducing the T and altering the $\rho$ of another valve path in which the

$T_2$ is greater. We have found this quite feasible practically *.

It still remains to consider how emission changes will affect such a "lumped characteristic." For continued balance, despite emission changes, of say grid and anode currents, we see from equation (II.) that each of the terms on the right side for $I_A$ should match the corresponding ones for $I_S$. This can usually be achieved provided that the values of $\frac{dI}{de}$ ( and $\frac{d\rho}{de}$ ) in the two circuits

Fig. 7.

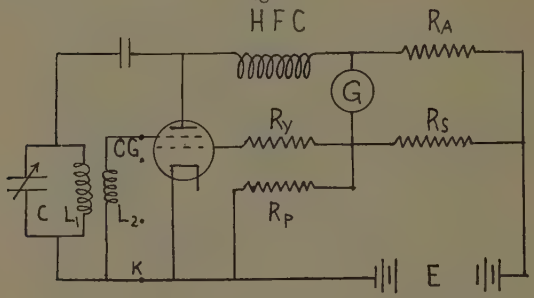
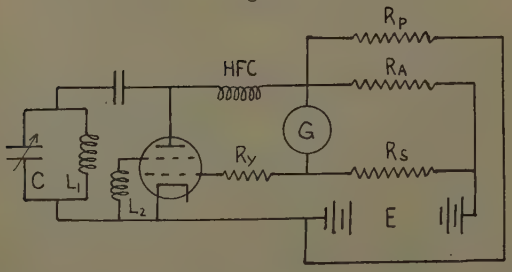



Fig. 8.




vary in proportion. It is reasonable to expect this to occur.

The embodiment of these principles in a practical form yields a singularly simple circuit (figs. 7 and 8),

* From this equation it appears that $\mu \, . \, dVg = dT$, which suggests that T may also be altered by biassing the grid.

which, at first sight, bears a close resemblance to Du Bridge's. It must be reiterated, however, that this circuit of ours employs ordinary directly heated valves,

SCHEDULE A.

Ultramicrometer circuit (fig. 8) Osram DG 2 valve, U.M. capacity C=1000 cm. In balanced circuit galvanometer (300 divs. per microampere) gives deflexion of 600 divisions for 1 cm. change in C. Preliminary balancing effected using microammeter until 5 per cent. change in battery gives only a fractional deflexion.

$R_A$=450 ohms.

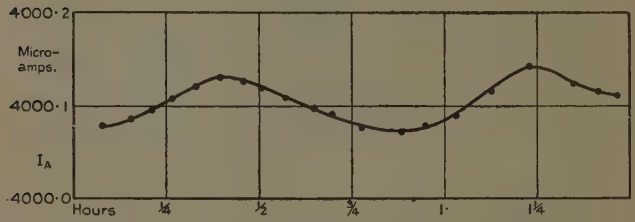
| $R_y$ ohms. | $R_p$ ohms. | $R_s$ ohms. | $I_A$ ma. | $I_S$ ma. | Galvanometer deflexions fsr 5 per cent. change. | |
|---|---|---|---|---|---|---|
| Galvanometer at zero. | | | (Need not be observed, quoted for information only.) | | of H.T. | of cathode. |
| 0 | 8850 | 256 | 3·7 | 9·6 | +1·0 | +55 |
| 200 | „ | 296 | 3·6 | 8·05 | −1·5 | +28 |
| 400 | „ | 336 | 3·5 | 7·0 | −5 0 | +13 |
| 600 | „ | 373 | 3·4 | 6·2 | −6·0 | +4·0 |
| 1000 | „ | 442 | 3·2 | 5·1 | −5·5 | −4·5 |
| 1100 | „ | 458 | „ | 4·9 | −5·0 | −5·0 |
| „ | 7850 | 479 | 3·1 | 4·85 | −4·0 | −4·0 |
| „ | 6850 | 505 | „ | 4·6 | −3·0 | −1·5 |
| 1200 | „ | 525 | „ | 4·55 | −2·75 | −2·5 |
| 1300 | 5850 | 567 | 3·05 | „ | −1·0 | ·0 |
| 1400 | „ | 604 | 3·00 | 4·2 | − ·5 | −1 |
| „ | 5350 | 631 | „ | „ | 0 | 0 |

operating at their usual high voltages. Furthermore, no special measures need be taken to obtain unusual constancy in the various batteries. Thus with an Osram DG 2 valve in the circuit shown the anode and space charge grid currents were three and four milliamperes respectively, while the (balanced) galvanometer sensitivity was 1/300 microampere per mm. division. With this arrangement we have readily obtained conditions of balance effective against more than 5 per cent. change in both cathode heating and H.T. battery voltages. In common with all other workers we find it necessary

to allow a period of some hours to elapse so that thermal changes in the valves have become stabilized. These give rise to disturbances which are not dealt with by the balance circuit.

To balance our circuit it is convenient to proceed according to the following schedule, which is an actual working example. In practice this process takes but little time and is quite straightforward. The preliminary stages are carried out with the galvanometer heavily shunted, or replaced by a microammeter. The necessary temporary changes in battery voltages are, of course, readily effected by switching in or out of suitable series resistances. These are not shown in the diagram (fig. 8) *.

Fig. 9.




The above circuit was switched on each day for several days, and on running for about an hour and a half the galvanometer came to rest within two or three centimetres of the same spot. The circuit was then tested and found to be still in perfect balance against battery variations. On the fourth day a long run of observations (fig. 9) was taken. We may point out that no steps were taken to shield the neighbouring parts of the circuit from the thermal effects of the valve and that the standard (Sullivan) condenser, together with the rest of the set, were also subject to room temperature variations. As a change of two parts per million of the capacity would deflect the galvanometer through more than 1 mm. it is reasonable to suppose that the residual slow drifting shown is due to such thermal effects.

February 1st, 1935.

* It will be noticed that $R_p$ is here shunting the *anode circuit*.

## XXXVIII. *Notices respecting New Books.*

*The Diary of Robert Hooke, M.A., M.D., F R.S.* 1672–1680. Edited by HENRY W. ROBINSON, Librarian of the Royal Society, and WALTER ADAMS, B.A. With a Foreword by Sir F. G. HOPKINS, O.M., President of the Royal Society. [Pp. xxviii+527.] (London : Messrs. Taylor & Francis, 1935. Price 25*s*.)

ALTHOUGH the manuscript of Hooke's Diary has been in the Guildhall for forty-four years, it has remained practically unknown until recently. Dr Waller, who had published a life of Hooke about 1705, received the MS. amongst other papers, and probably intended to deal with it in a further volume of posthumous works ; but on his death he had made very little progress. It then fell into the hands of a Rev. Wm. Derham. Nothing more was heard of it until it was offered for sale privately from a library at Moor Hall, Essex, in 1891, and was bought by the Corporation of the City of London. It came to the notice of the present Editors about seven years ago, and through the influence of Sir Frederick Hopkins (who has written a Foreword) permission was obtained to transcribe and publish it. The present volume is the result.

It is not merely a transcription, It includes a most interesting account of Hooke's early life. He was, from childhood, an inventive genius : he made a wooden clock that would "go" ; he painted pictures with home-made paints. On his father's death in 1648 he was sent to London to be apprenticed to Peter Lely, the fashionable portrait painter. He did not stay long with Lely, but became a pupil at Westminster School, where he made the acquaintance of Dr. Busby, who introduced him to geometry, and thence to mechanics—"his first and last mistress." Thence to Oxford, where he became assistant to the Hon. Robert Boyle. He contrived and perfected an air-pump for Boyle. The "experimentall philosophical clubbe" recognized Hooke's gifts, and it is clear that he was on terms of personal intimacy with them. He was then about 20 years old, while Boyle was 28, Christopher Wren 23, Ward 38, and Wilkins, the eldest of the group, was 41. In 1662 he was proposed as the Curator of experiments for the newly formed Royal Society, and it was ordered "that Mr. Hooke should come and sit amongst them and both bring in every day of the meeting three or four experiments of his own and take care of such others as should be mentioned to him by the Society." Thus began his association with the Society, which continued for forty years till his death. Other posts were added ; he became Professor of Geometry at Gresham College in 1664/5. "He proved to be one of the few contemporary Gresham Professors who faithfully discharged their duties."

Then came the plague, followed in 1666 by the Great Fire, which within four days destroyed the greater part of the City within the walls.

This opened a new phase in the career of Hooke. The King appointed Christopher Wren, Hugh May, and Roger Pratt to join with "such Surveyors and Artificers as should be appointed by the City" in connexion with the rebuilding: Hooke was appointed as one of these Surveyors at a salary of £150 per annum. For many years much of his time was taken up with these surveys for the New London.

When the Diary begins (March 10, 1671/2) Hooke was 37 years of age and well known. His 'Micrographia' was published in 1665. New evidence is brought forward that he had closely anticipated the theory "propounded by Lavoisier at the close of the eighteenth century, and still accepted as the fundamental doctrine of chemistry," in regard to the nature of combustion.

The Diary was kept for twelve years. In this volume a beginning has been made in August 1, 1672, and a finish at the end of 1680. The Diary gives a very good idea in outline of Hooke's activities. Many will be disappointed, however, at the exceedingly laconic character of many of the entries. Thus it is known that Newton had responded to an invitation of Hooke by sending in a suggestion to test the earth's diurnal motion by an experiment with falling bodies. This letter was read at the meeting of the Royal Society on Dec. 4, 1679 (see 'Isaac Newton,' by Louis T. More, p. 226). The Diary reports a full meeting of the Society on the 4th, but does not mention the letter, which was well received and discussed by Wren and Flamsteed. On the 5th, however, there is the note, "wrote to Mr. Newton." It is clear that much searching may be necessary before the full significance of many of the entries will be ascertained.

One of the chief contributions of the Editors consists of a Biographical Index of the individuals alluded to in the Diary. This covers more than 50 pages. Each character is identified when possible, thus :—

> Adams, John: an Inner Temple Barrister. The first to publish a map of England with London for the Prime Meridian; 301.

A list is also given of the Taverns and Coffee-houses mentioned by Hooke. We would like to know more about the discussions that took place therein.

Interest in the volume is enhanced by the illustrations, which include the Title Page of the 'Micrographia,' a Ballot List of the Royal Society (1671), a Facsimile of the Diary, Designs by Hooke, etc.

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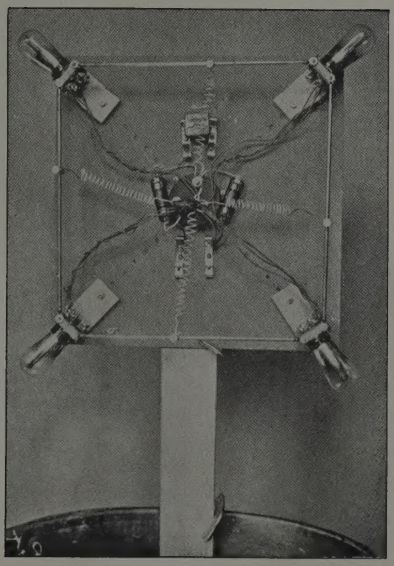
[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

FIG. 1.



Photograph of the laboratory transmitter.